Team Selection Test for RMM2025

1. Given an odd prime p and a positive integer b. A sequence c satisfies:

$$c_0 = 0, c_1 = 1 \text{ and } c_{n+2} - bc_n = 4(c_{n+1} - c_n) \text{ for all } n \geq 0$$

Prove that c_{p+1} is not divisible by p if and only if b is a quadratic residue modulo p (i.e. $b \equiv a^2 \pmod{p}$ for some $a \in \mathbb{Z}$).

D. Zmiakou

2. Given an integer $n > 0$. Numbers $1, 2, \ldots, n$ are written on the board in this order. Every move Lena erases two adjacent numbers and writes there their arithmetic mean. After $n-1$ moves only the number r was left on the board. Find all n for which r could be an integer.

M. Zorka

3. Let $n > 1$ be a positive integer. Prove that for any real numbers $a_0, a_1, \ldots, a_{n+1}$ the following equality holds

$$\sum_{i=1}^{n} (a_0 - a_n)(a_1 - a_n) \ldots (a_{i-1} - a_n) \cdot (a_{i+1} - a_1)(a_{i+2} - a_1) \ldots (a_{n+1} - a_1) = 0$$

D. Zmiakou

4. Circles ω and Ω are tangent internally at point T such that ω lies inside Ω. Tangent lines to ω from $A \neq T$ on Ω intersect Ω for the second time at points B and C, and $AB \neq AC$. Let M be the midpoint of BC, and I be the center of the incircle of the triangle ABC. Line MI intersects lines AB and AC at points P and Q respectively.

Prove that the circle which passes through T and I and tangent to MI, is tangent to the circumcircle of triangle APQ.

M. Zorka

First day of TST for IMO (Silk Road 2025)

1. A set S of real numbers is such that $1+\frac{1}{x} \in S$ for each $x \in S$. Is it possible that S contains exactly 2025 elements?

Prove that c_{p+1} is not divisible by p if and only if b is a quadratic residue modulo p (i.e. $b \equiv a^2 \pmod{p}$ for some $a \in \mathbb{Z}$).

2. Altitudes of an acute scalene triangle ABC meet at point H. Points M and N are the midpoints of the segments AB and CH, respectively. R is the foot of the perpendicular from H to line CM. T is the second intersection of the line MN with the circumcircle of triangle CNR. Let P be the circumcenter of the triangle formed by the lines CM, AH and BH. Prove that the lines HP and CT are perpendicular.

3. An odd integer $a > 1$ is given. Initially, Basil chooses an even positive integer b such that $b < a$ and tells it to Pete. Basil then writes down three integers on a blackboard. After that, Pete makes a sequence of moves. By a move, Pete can either add a to one of the numbers on the blackboard, add b to the second number, and subtract $a+b+1$ from the third number, or conversely subtract a from one number on the blackboard, subtract b from the second number, and add $a+b+1$ to the third one. At each move, Pete can independently choose which number on the blackboard is the first, the second, and the third. Pete wins if, after some moves, all the three numbers on the blackboard are zero. For which a Basil cannot prevent Pete's win?

4. Let $p > 200$ be a prime number. We call a positive integer n good if p divides the numerator of the irreducible fraction $\frac{a_n}{b_n} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Prove that for all large enough N the number of good numbers not exceeding N is not greater than $CN^{\frac{3}{4}}$, where C is a constant possibly depends on p.

Second day of TST for IMO

5. Prove that for any positive integer n the following equality holds

$$n = \sum_{k=0}^{2n-1} \left[\frac{2025(2k+1)}{4n} + \frac{1}{2} \right] - \sum_{k=0}^{2n-1} \left[\frac{2025(2k+1)}{4n} \right].$$

D. Zmiakou

6. We call a polynomial with integer coefficients interesting if it attains the value 2 at exactly two integer points, and attains the value 10 at exactly four integer points. Over all interesting polynomials of minimal degree, find all possible values of the minimal value of the polynomial over real numbers.

M. Karpuk

7. Find the number of sequences $c_1, c_2, \ldots, c_{2n}$ of integers such that $c_{i+n} = c_i$ for all $1 \le i \le n$, $c_1 \ge -1$, $c_n = 0$, and for all $1 \le i < j \le 2n$ such that $j \le i + n - 1$ the following inequality holds

$$i - j \le c_j - c_i \le 1.$$

M. Zorka

8. Let T be the intersection of diagonals AC and BD of a cyclic quadrilateral $ABCD$. Let the foots of perpendiculars from T to AB, BC, CD, AD be P, Q, R, S. Let X be the second intersection of the circumcircle of triangle PQS and line QT, Y be the second intersection point of the circumcircle of triangle PRS and line RT, Z be the second intersection of the circumcircle of triangle QPR and line PT, W be the second intersection of the circumcircle of triangle SQR and line ST. a) Prove that $XY \parallel ZW$. b) Prove that lines XZ, YW and AC either concur, or are pairwise parallel.

A. Mialik

Third day of TST for IMO

9. In a scalene triangle ABC with altitude CH, let $AB = c, BC = a, AC = b, CH = h, \angle C = \gamma$. Construct a triangle $A_1B_1C_1$ with side lengths $a + b, c + h \cdot ctg(\frac{\gamma}{2}), h \cdot ctg(\frac{\gamma}{2})$ and $A_2B_2C_2$ with side lengths $|a - b|, |c - h \cdot tg(\frac{\gamma}{2})|, h \cdot tg(\frac{\gamma}{2})$. Prove that one of the angles of triangle $A_1B_1C_1$ is equal to one of the angle of triangle $A_2B_2C_2$.

D. Bazyleu

10. Given a circumscribed quadrilateral $ABCD$. A point P lies on the side AB. The circumcircle ω of triangle BCP intersects DP for the second time at point Q. Let E be the reflection of B in P. It turned out that points A, D, E, Q are concyclic. Prove that the tangent line to ω at P passes through the midpoint of AD.

M. Zorka

11. Given an not strictly increasing function $f : \mathbb{N} \to \mathbb{N}$. It is known that for any positive integers $x \neq y$ the following inequality holds

$$f(x) + f(y) \leq f(f(x) + f(y)) + f(|x - y|)$$

Prove that there exists a positive integer M such that $f(n) \leq n$ for all $n \geq M$.

M. Zorka

12. The teacher wrote an arbitrary positive integer on the board. A student Misha in one move can do the following operation: choose a number on the board, and then either multiply it by 3 and then add 1,2,4 or 8, or divide it by 2026. The results are then written on the board. Prove that Misha can achieve 1 in finitely many moves.

D. Zmiaikou

Fourth day of TST for IMO

13. Given a $n \times m$ checkered board. In one move Matvey can remove any edge, and the symmetrical with respect to the center of the board to it edge. It turned out that after k moves there were no cyclic paths on the board. Find the minimal possible value of k.

D. Zmiaikou

14. Find all positive integers n with even number of divisors which can be paired such that the sum of numbers in every pair is a power of three.

M. Zorka

15. In a regular hexagon H 100 regular unit hexagons were placed. It turned out that any line parallel to a side of H intersects the interior of at most two unit hexagons. Find the minimal possible value of side length of H.

D. Zmiaikou

16. For a polynomial Q with rational coefficients with prime degree q and polynomial $P(x, y)$ with rational coefficients and a smaller degree

$$P(x, y) - P(y, x) \mid Q(x) - Q(y)$$

a) Prove that $P(x, y) - P(y, x)$ is a polynomial of degree 1. b) Find all prime numbers q for which the statement of a) will be true if the coefficients of P and Q can be real numbers.

M. Karpuk

Fifth day of TST for IMO

17. Let the incircle with center I of triangle ABC touch BC at T. Points K, W on arc BAC of circumcircle of triangle ABC are such that $\angle IKA = 90°$, $BW = WC$. Let WT intersect the circumcircle of triangle ABC for the second time at L. Let L' be the reflection of L in C. Prove that the circumcircle of triangle KCL' is tangent to BC.

M. Zorka

18. A sequence of positive integers $a_1, a_2, \ldots$ is such that $|a_{n+2} - a_{n+1}| \le |a_{n+1} - a_n|$ for all positive integers n. It is known that for any positive integer k, there exist indices $i_1, \ldots, i_k, j_1, \ldots, j_k$ and positive integer n such that $a_{i_r} = (n + r)a_{j_r}$ for all $r = 1, \ldots, k$. Prove that there exists m such that for any $n > m$ there are i, j such that $a_i = 3^n a_j$.

M. Zorka

19. Find all prime numbers p such that $3^p - 1 \,\vdots\, 4p + 1$.

V. Kamianetski, Y. Sheshukou

20. Given positive integers d, n. Find all real numbers x such that

$$\sum_{k=0}^{n} \frac{(-1)^k (x-k)^d}{k!(n-k)!} \ge 0$$

D. Zmiaikou