50 Belarusian Mathematical Olympiad

БЕЛАРУСКАЯ МАТЭМАТЫЧНАЯ ШКОЛЬНІКАЎ АЛІМПІЯДА

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БРЭСТ 2000

Minsk
2000

The STRUCTURE of *BELARUSIAN MATHEMATICAL OLYMPIADS*

The olympiad is divided into four rounds:

1st – school round (October)
2nd – district round (November)
3rd – regional round and Minsk Mathematical Olympiad (January)
4th – final round (March – April)

The third and fourth rounds take two days
(usually 4 problems for 4 hours)

All participants are divided into four categories in accordance with the forms they study at:

Category A – 11 form (16 – 17 years)
Category B – 10 form (15 – 16 years)
Category C – 9 form (14 – 15 years)
Category D – 8 form (13 – 14 years)

The 50th Belarusian Mathematical Olympiad was prepared by
Dr. E.Barabanov, Dr. V.Kaskevich, Dr. S.Mazanik, Dr. I.Voronovich, Dr. I.Zhuk

Selection and Training session to prepare Belarusian Team for 41th IMO was held at the Belarusian State University in several stages (April – June) with I.Voronovich, S.Mazanik and V.Kaskevich as Belarusian team's coaches.

The following students were selected to represent Belarusian Team in 41th IMO:
Yury Hrushetski, Dzmitry Doryn, Aliaksandr Kirkouski,
Artsiom Kokhan, Siarhei Markouski, Alexandr Usnich

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PROBLEMS

FINAL ROUND

Category D

First Day

1. Find all pairs of integer numbers (x, y) satisfying the equality

$$3xy - x - 2y = 8.$$

(U. Menski)

2. Points M and K are marked on the sides BC and CD of the square $ABCD$, respectively. Let P be the intersection point of the segments MD and BK.

Prove that AP and MK are perpendicular if $MC = KD$.

(S.Mazanik)

3. The roots of the quadratic equation

$$ax^2 - 4bx + 4c = 0, \quad a > 0,$$

are on the segment $[2; 3]$.

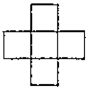
Prove that

a) $a \leq b \leq c < a + b$;

b) $\dfrac{a}{a+c} + \dfrac{b}{b+a} > \dfrac{c}{c+b}$.

(S.Mazanik)

4. There is an 8×8 square board divided with lines parallel to its sides into 64 smaller congruent 1×1 squares. The cross-shaped tiles (see the fig.) are placed on the board such that the borders of the tiles pass along the sides of the squares and the tiles overlap neither each other nor the edges of the board.



Determine the largest possible number of the tiles that can be placed on the board.

(I. Zhuk, V. Kaskevich)

Second Day

5. In the final round of the mathematical olympiad Problem 5 of category D was worth 4 points. After the olympiad was over it turned out that the number of students scoring 3 points on this problem was equal to the number of students scoring 2 points. Each student scored at least 1 point for this problem.

Determine the number of the students scoring at least 3 points on Problem 5, if the total number of the points for this problem are 30 greater than the number of the participants of category D.

(V.Kaskevich)

6. Let $f(x) = \{x\} + \left\{\frac{1}{x}\right\}$, where $\{x\}$ is the fractional part of x.

a) Prove that $f(x) < 1,5$ for $x > 0$ and $f(x) < 2$ for $x < 0$.

b) Prove that for any positive integer n there exists x_0 such that

$$f(x_0) > 2 - \frac{1}{n}.$$

(I.Gorodnin)

7. Points B_1 and C_2 are marked on the side AB of a triangle ABC ($BC < AC < AB$) such that $AC_2 = AC$, $BB_1 = BC$. Point B_2 is marked on AC such that $CB_2 = CB$, and C_1 is marked on the extension of the side CB (B lies between C and C_1) such that $CC_1 = CA$.

Prove that the lines C_1C_2 and B_1B_2 are parallel.

(I.Voronovich)

8. Seven points are marked on a plane, no three of them lie on the same straight line. Any two points are connected with a segment.

Is it possible to color these segments by some colors so that there are exactly three segments of any color, all three forming a triangle with vertices at the marked points?

(U.Mische)

Category C

First Day

1. Find all pairs of integer numbers (x, y) satisfying the equality

$$y(x^2+36)+x(y^2-36)+y^2(y-12)=0.$$

(I. Voronovich, S. Mazanik)

2. Let F be the intersection point of the altitude CD and the bisector AE of the right-angled triangle ABC ($\angle C = 90^0$). Let G be the intersection point of ED and BF.

Prove that the area of the quadrilateral $CEGF$ is equal to the area of the triangle BDG.

(I. Zhuk)

3. Set S consists of k sequences each having the length n with terms from the set $\{0, 1, 2\}$. For any two of the sequences $(a_i), (b_i)$ from S one can construct the sequence (c_i) such that

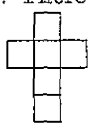
$$c_i = \left[\frac{a_i+b_i+1}{2}\right], \quad i = 1, \ldots, n,$$

($[x]$ is the greatest integer not exceeding x) and include this sequence into the set S, and so on. It is known that from initial set S one can obtain the set with 3^n different sequences. (Two sequences are different if they differ in at least one term.)

Find the smallest possible value of k.

(P. Lukianenko)

4. There is an 9×9 square board divided with lines parallel to its sides into 81 smaller congruent 1×1 squares. The cross-shaped tiles (see the fig.) are placed on the board such that the borders of the tiles pass along the sides of the squares and the tiles overlap neither each other nor the edges of the board.



Determine the largest possible number of the tiles that can be placed on the board.

(I. Zhuk, E. Barabanov, V. Kaskevich)

Second Day

5. Determine the total number of pairs of positive integers (p, q) such that the roots of the equation $x^2 - px - q = 0$ are not exceed 10.

(M.Naumik)

6. The equilateral triangles $\triangle ABF$ and $\triangle CAG$ are constructed externally on the hypotenuse AB and the leg CA of the right triangle ABC. Let M be the midpoint of BC.

Find the length of BC if $MF = 11$ and $MG = 7$.

(I.Voronovich)

7. Tom and Jerry play the following game. They, in turn (Tom is the first), put pawns into cells of 20×20 square board. Per move it is allowed to put the pawn into the empty cell. The player wins if after his move some four pawns are the vertices of a rectangle with sides parallel to the sides of the board.

Who of the players wins if both of them play to win?

(V.Kaskevich)

8. Given that the numbers a, b, c, d satisfy the equation

$$\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{d} + \frac{d}{c} + \frac{d}{a} + \frac{a}{d} = \frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b} + \frac{ac}{bd} + \frac{bd}{ac} + 2,$$

prove that at least two of them are equal.

(V.Kaskevich)

Category B

First Day

1. Find the locus of points M on the Cartesian plane Oxy such that the tangents from M to the parabola $y = x^2$ are perpendicular.

(I. Gorodnin)

2. Find all pairs of positive integer numbers (m, n) satisfying the equality

$$(m - n)^2(n^2 - m) = 4m^2n.$$

(I. Voronovich)

3. Let M be the intersection point of the diagonals AC and BD of a convex quadrilateral $ABCD$. Let K be the intersection point of the extension of the side BA over A and the bisector of the angle ACD.

If $MA \cdot MC + MA \cdot CD = MB \cdot MD$, prove that $\angle BKC = \angle CDB$.

(S. Shikh)

4. An equilateral triangle of side n is divided into n^2 equilateral triangles of side 1 by lines parallel to the sides of the triangle. Each point that is a vertex of at least one of these unit triangles is labelled with a number; exactly one of these points is labelled with -1, all the others with 1's. Per move one can choose a line passing through the side of one of the small triangle and change the signs of numbers for all labelled points on this line.

Determine all possible situations (the value of n and the position of -1) for which one can obtain the arrangement with all 1's from the initial arrangement using the described operations.

(I. Zhuk)

Second Day

5. Tom and Jerry play the following game. They, in turn (Tom is the first), put pawns into cells of 25×25 square board. Per move it is allowed to put the pawn into the empty cell. The player wins if after his move some four pawns are the vertices of a rectangle with sides parallel to the sides of the board.

Who of the players wins if both of them play to win?

(V.Kaskevich)

6. A rectangle $ABCD$ and a point X are on a plane.

a) Prove that one can choose from the segments XA, XB, XC, XD some three which are the sides of a triangle.

b) Is the previous statement true, if $ABCD$ is a parallelogram?

(I.Voronovich)

7. Find all positive integers a and b satisfying the equality $a^{(a^a)} = b^b$.

(I.Voronovich)

8. The set R of non-zero vectors on a plane is called *concordant*, if it satisfies the following conditions:

1) for any vectors $\vec{a}$ and $\vec{b}$ (may be, equal) of R the vector $S_{\vec{b}}(\vec{a})$ which is symmetric to $\vec{a}$ with respect of the straight line l, $l \perp \vec{b}$, belongs to R;

2) for any $\vec{a}, \vec{b} \in R$ there exists some integer $k = k(\vec{a}, \vec{b})$ such that $\vec{a} - S_{\vec{b}}(\vec{a}) = k\,\vec{b}$.

a) Prove that for any $\vec{a}, \vec{b} \in R$, $\vec{a} \nparallel \vec{b}$, $\vec{a} \not\perp \vec{b}$, either $\vec{a} - \vec{b} \in R$, or $\vec{a} + \vec{b} \in R$.

b*) Does there exist an infinite concordant set? Find the largest possible value of non-zero vectors in a finite concordant set.

(A.Mirotin, E.Barabanov)

Category A

First Day

1. Pit and Bill play the following game. At the beginning, Pit writes on the blackboard a number a, then Bill writes a number b, and then Pit writes a number c. Can Pit choose his numbers in such a way that the following three equations

$$\begin{gathered} x^3 + ax^2 + bx + c = 0, \\ x^3 + bx^2 + cx + a = 0, \\ x^3 + cx^2 + ax + b = 0 \end{gathered}$$

have

a) a common real root?

b) a common negative root?

(S. Sobolevski)

2. How many pairs (n, q) satisfy the equality

$$\{q^2\} = \left\{\frac{n!}{2000}\right\}$$

with n positive integer, q noninteger rational number, $0 < q < 2000$?

(S.Shikh)

3. There is a sequence $e_1, e_2, \ldots, e_N$, $e_i \in \{-1, 1\}$, $i = 1, 2, \ldots, N$, $N \geq 5$. Per move one can choose any five consecutive terms and change the signs of chosen numbers. Two sequences are said to be *similar* if one of them can be obtained from the other with a finite number of the described operations.

Find the maximal number of the sequences no two of which are similar to each other.

(E. Barabanov)

4. A line l intersects lateral sides and diagonals of a trapezoid. It is known that the segment of l between the lateral sides is divided by the diagonals into three equal parts.

Does it follow that the line l is parallel to the bases of the trapezoid?

(E. Barabanov)

5. Nine points are marked on a plane, no three of them lie on the same straight line. Any two points are connected with a segment.

Is it possible to color these segments by some colors so that there are exactly three segments of any color, all three forming a triangle with vertices at the marked points?

(U.Mische)

6. A vertex of a triangular pyramid is called *perfect*, if one can construct a triangle with edges from this vertex as its sides.

How many perfect vertices can the triangle pyramid have? (Determine all possible cases.)

(E.Barabanov)

7. a) Find all positive integers n such that the equation $(a^a)^n = b^b$ has at least a solution in positive integers a, b exceeding 1.

b) Find all positive integers a and b satisfying the equality $(a^a)^5 = b^b$.

(I.Voronovich)

8. To any $\triangle ABC$ with $AB = c\,(m)$, $BC = a\,(m)$, $CA = b\,(m)$ and $\angle A = \alpha\,(rad)$, $\angle B = \beta\,(rad)$, $\angle C = \gamma\,(rad)$ we assign the set of numbers $(a, b, c, \alpha, \beta, \gamma)$.

Find the minimal value of n for which there is a non-isosceles $\triangle ABC$ such that there are exactly n distinct numbers in the set $(a, b, c, \alpha, \beta, \gamma)$.

(I.Loseu)

SELECTION and TRAINING SESSION

Test 1 (*Time: 4 hours*)

1. Find the minimal number of cells on the 5×7 board that must be painted so that any cell which is not painted has exactly one painted neighboring cell. (We call two different cells neighboring if they have a common side.)

(E.Barabanov, I.Voronovich)

2. Let P be a point inside the right triangle ABC, $\angle C = 90^0$, such that $AP = AC$, M be the midpoint of the hypotenuse AB, H the foot of the altitude CH.

Prove that PM is a bisector of $\angle BPH$ if and only if $\angle A = 60^0$.

(Antonio Caminha, Brasil)

3. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n-1)) = f(n+1) - f(n)$$

for any $n \geq 2$?

(I.Voronovich)

4. A closed pentagonal line is inscribed in a sphere of the diameter 1. All edges of the line are l. Prove that $l \leq \sin 72^0$.

(A.Korenev)

Test 2 (*Time: 4 hours*)

1. All the vertices of a convex polyhedron have the same degree 4. Find the minimal possible number of triangular faces in the polyhedron.

(V.Kaskevich)

2. Real numbers a, b, c satisfy the equation

$$2a^3 - b^3 + 2c^3 - 6a^2b + 3ab^2 - 3ac^2 - 3bc^2 + 6abc = 0.$$

Given that $a < b$, determine which of b and c is larger.

(I.Voronovich)

3. We call two integer points (a_1, b_1) and (a_2, b_2) on the coordinate plane *connected*, if either $a_2 = -a_1$ and $b_2 = b_1 \pm 1$, or $a_2 = a_1 \pm 1$ and $b_2 = -b_1$.

Prove that it is possible to construct an infinite sequence (m_1, n_1), ..., (m_k, n_k), ... of integer points such that any two neighboring points of the sequence are connected, and each integer point of the plane occurs in this sequence.

(D.Zmeikov)

4. In a triangle ABC with $AC = b$, $CB = a$ $(a \neq b)$ points E and F are on the sides AC and BC respectively, such that $AE = BF = \dfrac{ab}{a+b}$. Let M be the midpoint of AB, N the midpoint of EF, P the intersection point of the segment EF and the bisector of $\angle ACB$.

Find the ratio of the areas $S(CPMN)/S(ABC)$.

(I.Loseu)

Test 3 (*Time: $4\frac{1}{2}$ hours*)

1. For a triangle ABC, let $a = BC$, $b = AC$; m_a, m_b be the medians from the vertices A, B, respectively.

Find all real λ such that the equality $m_a + \lambda a = m_b + \lambda b$ implies that ABC is isosceles, i.e. $a = b$.

(I.Voronovich)

2. a) Prove that $\{n\sqrt{3}\} > \dfrac{1}{n\sqrt{3}}$ for any positive integer n. (Here $\{x\}$ denotes the fractional part of x.)

b) Does there exist a constant $c > 1$ such that $\{n\sqrt{3}\} > \dfrac{c}{n\sqrt{3}}$ for any positive integer n?

(I.Voronovich)

3. A graph has 15 vertices. Each edge of the graph is colored either red or blue such that there is no three vertices A, B, C connected pairwise with edges of the same color.

Determine the largest possible number of the edges of this graph.

(D.Badziahin)

Test 4 (*Time: $4\frac{1}{2}$ hours*)

1. Find all functions f, g, $h : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y^3) + g(x^3 + y) = h(xy)$$

for all real x and y.

(I.Voronovich)

2. Let ABC be a triangle and M be an interior point. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + AC + BC.$$

(IMO-99 Shortlist)

3. Prove that for any positive integer N there exist positive integers a_0 and d, where d is not divisible by 2 and 5, such that for any number a_k of the arithmetic sequence $a_k = a_0 + kd$, $k \in \mathbb{N}\bigcup\{0\}$, the sum $S(a_k)$ of digits in the decimal representation of a_k is greater than N.

(S.Shikh, IMO-99 Shortlist)

Test 5 (*Time: $4\frac{1}{2}$ hours*)

1. Let AM and AL be the median and the bisectrix of the triangle ABC (M, $L \in BC$); $\angle BAC = \alpha$, $BC = a$, $AM = m_a$, $AL = l_a$.

Prove the inequalities:

a) $a \tan\dfrac{\alpha}{2} \le 2m_a \le a\cot\dfrac{\alpha}{2}$, if $\alpha < \dfrac{\pi}{2}$ and $a\cot\dfrac{\alpha}{2} \le 2m_a \le a\tan\dfrac{\alpha}{2}$, if $\alpha > \dfrac{\pi}{2}$;

b) $2l_a \le a\cot\dfrac{\alpha}{2}$.

(P.Lukianenko)

2. Let n, k be positive integers such that n is not divisible by 3 and $k \ge n$. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in decimal representation is k.

(IMO-99 Shortlist)

3. Suppose that every integer has been given one of the colors red, blue, green or yellow. Let x and y be odd integers so that $|x| \neq |y|$. Show that there are two integers of the same color whose difference has one of the following values: $x, y, x+y$ or $x-y$.

(IMO-99 Shortlist)

Test 6 (*Time: $4\frac{1}{2}$ hours*)

1. Let $M = \{1, 2, ..., 39, 40\}$. Find the smallest n, $n \in \mathbb{N}$, for which it is possible to partition M into n subsets M_1, M_2, ..., M_n so that none of M_i's contains elements a, b, c such that $a = b + c$ (a, b, c not necessarily different.)

(U.Menski)

2. A positive integer $m = \overline{a_k ... a_1 a_0}$ is called *monotonic* if

$$a_k \leq a_{k-1} \leq ... \leq a_0.$$

Prove that for any $n \in \mathbb{N}$ there exists an n-digit monotonic number which is a perfect square.

(T.Lasy)

3. Starting with an arbitrary pair $(\vec{a}, \vec{b})$ of vectors on the plaine we are allowed to perform the operations of the following two types:

A. To change $(\vec{a}, \vec{b})$ on $(\vec{a} + 2k\vec{b}, \vec{b})$, where k is an arbitrary nonzero integer.

B. To change $(\vec{a}, \vec{b})$ on $(\vec{a}, \vec{b} + 2k\vec{a})$, where k is an arbitrary nonzero integer.

Besides, we must change the type of operation on any step.

a) Is it possible to obtain the pair of vectors $((1,0),(2,1))$ from the pair $((1,0),(0,1))$ using the operations described, if we start with the operation of the type A?

b) Find all pairs $((a,b),(c,d))$, that can be obtained from $((1,0),(0,1))$, if we can arbitrarily choose the type of the first operation.

(A.Korenev)

Test 7 (*Time: $4\frac{1}{2}$ hours*)

1. Prove that for any positive real numbers a, b, c, x, y, z the inequality

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\geq\frac{(a+b+c)^3}{3(x+y+z)}.$$

(I.Gorodnin)

2. Points A, B, C divide the circumcircle Ω of the triangle ABC into three arcs. Let X be a variable point on the arc AB and O_1, O_2 be the incenters of the triangles CAX and CBX. Prove that the circumcircle of the triangle XO_1O_2 intersects Ω in a fixed point.

(IMO-99 Shortlist)

3. A game is played by n girls ($n \geq 2$), everybody having a ball. Each of the $\binom{n}{2}$ pairs of players, in an arbitrary order, exchange the balls they have at that moment. The game is called *"nice"* if at the end nobody has her own ball and it is called *"tiresome"* if at the end everybody has her initial ball. Determine the values of n for which there exists a nice game and those for which there exists a tiresome game.

(IMO-99 Shortlist)

Test 8 (*Time: $4\frac{1}{2}$ hours*)

1. Let P be the intersection point of the diagonals AC and BD of the convex quadrilateral $ABCD$ in which $AB = AC = BD$, and let O, I be the circumcenter and incenter of the triangle ABP, respectively.

Prove that OI and CD are perpendicular, if $O \neq I$.

(I.Voronovich)

2. Prove that there exists two strictly increasing sequences (a_n) and (b_n) such that $a_n(a_n+1)$ divides b_n^2+1 for every natural n.

(IMO-99 Shortlist)

3. Prove that the set of positive integers can not be partitioned into three non-empty subsets, such that for any two integers x, y, taken from any two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.

(I.Voronovich, IMO-99 Shortlist)

H I N T S and S O L U T I O N S

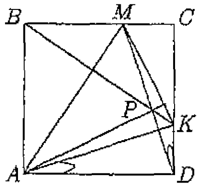
F I N A L   R O U N D

Category D

1. Answer: $(-8,0)$, $(0,-4)$, $(1,9)$, $(5,1)$.

We rewrite the initial equation as $(3x-2)(3y-1)=26$. So, $3x-2=m$, $3y-1=n$, where the integers m and n divide 26. From $3x-2 \equiv 1 \pmod 3$ it is easy to see that $m = -26, -2, 1$ or 13 (then $n=-1, -13, 26$, and 2 respectively). Now we find all possible values of x and y: $x=-8$, $y=0$; $x=0$, $y=4$; $x=1$, $y=9$; $x=5$, $y=1$.

2. Connect K with K and M. It is easy to see that $\Delta\, AKD = \Delta\, DMC$ since $AD = DC$, $DK = CM$, $\angle ADK = \angle DCM = 90^0$. Therefore, $\angle KAD = \angle MDC$, so $MD \perp AK$. Similarly, $KB \perp AM$. Hence P is the orthocenter of $\Delta\, AMK$, which gives $AP \perp MK$.



3. a) Let x_1 and x_2 be the roots of the given equation. By Vieta's formula $\frac{4b}{a} = x_1 + x_2 \geq 2+2=4$, so $b \geq a$. Again, by Vieta's formula $\frac{4c}{a} = x_1 \cdot x_2$, therefore, the required inequalities $b \leq c < a+b$ are equivalent to the inequalities $\frac{4b}{a} \leq \frac{4c}{a}$, $\frac{4c}{a} < \frac{4b+4a}{a} = \frac{4b}{a}+4$, or

$$x_1 + x_2 \leq x_1 \cdot x_2 < 4 + (x_1+x_2) \iff 0 \leq x_1 \cdot x_2 - (x_1+x_2) < 4. \qquad (1)$$

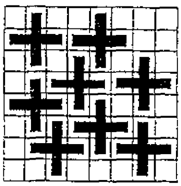
Since $x_1 \cdot x_2 - (x_1+x_2) = (x_1-1)(x_2-1)-1$, we see that (1) is equivalent to $1 \leq (x_1-1)(x_2-1) < 5$. The last inequalities hold since $x_1, x_2 \in [2,3]$.

b) From what has been already proved, it follows that

$$\frac{a}{a+c} + \frac{b}{b+a} \geq \frac{a}{b+c} + \frac{b}{b+c} = \frac{a+b}{b+c} > \frac{c}{b+c}.$$

4. Answer: 8.

It is easy to see that at most two tiles are adjacent to each side of the board. Therefore, at most 8 cells on the border of the square are covered by the tiles. So at most 44 cells can be covered by the tiles (8 are on the border of the square and $6 \cdot 6 = 36$ are inside the square). Since every tile consists of 5 cells we have at most 8 tiles on the board ($\frac{44}{5} < 9$). An example of exactly 8 tiles is shown in the figure.



5. Answer: 10.

Let x students score 1 point for Problem 5, y students score 2 points (the same number of students receive 3 points), z students score 4 points. Therefore the total number of participants is $x+2y+z$ and they together score $x+5y+4z$ points. By condition, $(x+5y+4z)-(x+2y+z)=30$, i.e. $3y+3z=30$. Hence the number of students having at least 3 points is equal to $y+z=10$.

6. a) For any real x we have $0 \le \{x\} < 1$, so the inequality $f(x) < 2$ holds for any x and, in particular, for $x < 0$.

Consider the case when $x > 0$. Let $x \ge 1$. Set $x = n+\alpha$, where $n \in \mathbb{N}$, $0 \le \alpha < 1$. Then $\{x\} = \alpha$ and $\left\{\frac{1}{x}\right\} = \frac{1}{n+\alpha}$. Therefore $f(x) = \alpha + \frac{1}{n+\alpha} \le \alpha + \frac{1}{1+\alpha}$. Show that $\alpha + \frac{1}{1+\alpha} < 1,5$ for any α, $0 \le \alpha < 1$. Indeed,

$$\alpha + \frac{1}{1+\alpha} < 1,5 \quad \Leftrightarrow \quad 2(\alpha^2+\alpha+1) < 3(\alpha+1) \quad \Leftrightarrow$$

$$2\alpha^2 - \alpha - 1 < 0 \quad \Leftrightarrow \quad (\alpha-1)(2\alpha+1) < 0.$$

The last inequality is obvious for any α, $0 \le \alpha < 1$.

So, the required inequality is proved for $x \ge 1$. To prove the inequality for $0 < x < 1$ it suffices to replace x by $\frac{1}{y}$, because

$$f(x) = \{x\} + \left\{\frac{1}{x}\right\} = \left\{\frac{1}{y}\right\} + \{y\},$$

and for $0 < x < 1$ we have $y > 1$.

b) By the above it follows that for $n \ge 2$ the desired x_0 must be negative. Set $x_0 = -\frac{m}{m^2+1}$, $m \in \mathbb{N}$. Then $\frac{1}{x_0} = -m - \frac{1}{m}$. So $\{x_0\} = 1 - \frac{m}{m^2+1}$ and $\{\frac{1}{x_0}\} = 1 - \frac{1}{m}$. Hence

$$\{x_0\} + \{\frac{1}{x_0}\} = 2 - \frac{m}{m^2+1} - \frac{1}{m} > 2 - \frac{1}{m} - \frac{1}{m} > 2 - \frac{1}{n} \quad \forall m > 2n,$$

which proves the required statement for any $n \ge 1$.

7. If $\vec{a} = \overrightarrow{BC}$, $\vec{b} = \overrightarrow{CA}$, then $\overrightarrow{BA} = \vec{a}+\vec{b}$. Let $a = |\vec{a}|$, $b = |\vec{b}|$, $c = |\vec{a}+\vec{b}|$, then $a < b < c$. By condition it follows that

$$\overrightarrow{C_1B} = \frac{b-a}{a}\vec{a},\ \overrightarrow{BC_2} = \frac{c-b}{c}(\vec{a}+\vec{b}),\ \overrightarrow{AB_2} = -\frac{b-a}{b}\vec{b},\ \overrightarrow{B_1A} = \frac{c-a}{c}(\vec{a}+\vec{b}).$$

Therefore

$$\overrightarrow{B_1B_2} = \overrightarrow{B_1A} + \overrightarrow{AB_2} = \frac{c-a}{c}\vec{a} + \left(\frac{c-a}{c} - \frac{b-a}{b}\right)\vec{b} = \frac{c-a}{c}\vec{a} + \frac{a(c-b)}{bc}\vec{b},$$

and

$$\overrightarrow{C_1C_2} = \overrightarrow{C_1B} + \overrightarrow{BC_2} = \left(\frac{b-a}{a} - \frac{c-b}{c}\right)\vec{a} + \frac{c-b}{c}\vec{b} = \frac{b(c-a)}{ac}\vec{a} + \frac{c-b}{c}\vec{b}.$$

It is clear that $\frac{b}{a}\overrightarrow{B_1B_2} = \overrightarrow{C_1C_2}$, that is the vectors $\overrightarrow{B_1B_2}$ and $\overrightarrow{C_1C_2}$ are collinear, so the lines B_1B_2 and C_1C_2 are parallel.

8. Answer: it is possible.

The total number of the segments are $7 \cdot 6/2 = 21$. So there must be exactly 7 triangles of different colors. Since any marked point is the origin of two sides of the same triangle, two segments of some three colors start from any point. In particular each point is the vertex of exactly three triangles. Hence, to give an example of required coloring it suffices to construct the 7×7 table such that its columns correspond to the points, and its rows correspond to the triangles. In any row one must mark three cells corresponding to the points making the triangle. Since any point is the vertex of exactly three triangles, there must be exactly three marked cells in each column. The example is shown in the figure.

	1	2	3	4	5	6	7
1	×	×	×				
2	×			×	×		
3	×					×	×
4		×		×		×	
5		×			×		×
6			×		×	×	
7			×	×			×

Category C

1. Answer: $(0,0)$, $(0,6)$, $(-8,-2)$, $(1,4)$, $(4,4)$.

Rewrite the given equation: $yx^2 + (y^2 - 36)x + y(y-6)^2 = 0$. It is evident, that if $y = 0$, then $x = 0$ and the pair $(x, y) = (0, 0)$ satisfies the equation. If $y \neq 0$, then the initial equation can be considered as the quadratic equation with respect to x. So,

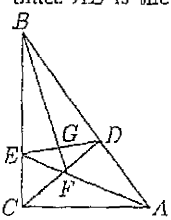
$$x = \frac{(36 - y^2) \pm \sqrt{(y^2 - 36)^2 - 4y^2(y-6)^2}}{2y}. \qquad (1)$$

For x to be integer the radicand must be a perfect square. But

$$(y^2 - 36)^2 - 4y^2(y-6)^2 = (y-6)^2((y+6)^2 - 4y^2) = 3(y-6)^2(6-y)(y+2).$$

Therefore, there is a positive integer m such that $3(6-y)(y+2) = m^2$. In particular, $(6-y)(y+2) \geq 0$, hence $-2 \leq y \leq 6$. It is easy to verify that only for $y = -2$, $y = 4$, and $y = 6$ the number $3(6-y)(y+2)$ is the perfect square. Considering these values of y, we find all required pairs: $(-8, -2)$, $(1, 4)$, $(4, 4)$ и $(0, 6)$.

2. Since AE is the bisectrix of $\triangle ABC$ and AF is the bisectrix of $\triangle ADC$, we have



$$\frac{CE}{BE} = \frac{AC}{AB} = \cos\angle BAC = \frac{AD}{AC} = \frac{DF}{CF} \implies$$
$$CE \cdot CF = BE \cdot DF = (BC - EC)(CD - CF) \implies$$
$$BC \cdot CD = BC \cdot CF + EC \cdot CD.$$

Multiplying both sides of this equality by $\frac{1}{2}\sin\angle BCD$ we obtain $S(BCD) = S(BCF) + S(ECD)$. But

$$S(BCD) = S(CEGF) + S(BEG) + S(BGD) + S(GDF),$$

$$S(BCF) = S(CEGF) + S(BEG), \quad S(ECD) = S(CEGF) + S(GDF).$$

which gives the required equality $S(CEGF) = S(BDG)$.

3. Answer : $n+1$.

It is evident that if the set contains 3^n various sequences of the length n, then it contains all possible sequences. From

$$c_i = \left[\frac{a_i + b_i + 1}{2}\right], \quad i = 1, \ldots, n,$$

it is easy to see that $c_i = 0$ only if $a_i = b_i = 0$, and $c_i = 2$ only if $a_i + b_i \geq 3$. Hence, the sequence containing only zeros (denote it by A_0), can not be obtain from the sequences different from A_0. So A_0 belongs to the initial set. Similar statements are valid for all sequences A_j with 2 at the j-th place, and with 0 at the other places. There are exactly n of such sequences. So $k \geq n+1$.

Show that from the sequences A_j, $j = 0, 1, \ldots, n$, one can obtain the set consisting of all 3^n possible sequences. Denote the operation of obtaining a new sequence by $\oplus$.

Also denote by $E(i_1, ..., i_k)$ the sequence with 1's at i_1-th, ..., i_k-th places. It is easy to see that

$$E(i_1, ..., i_k) = (...((A_{i_1} \oplus A_{i_2}) \oplus A_{i_3}) \oplus ... \oplus A_{i_k}).$$

Now denote by $F(i_1, ..., i_k; j_0)$ the sequence with 2 at j_0-th place and with 1's at i_1-th, ..., i_k-th places. We have

$$F(i_1, ..., i_k; j_0) = E(i_1, ..., i_k, j_0) \oplus A_{j_0}.$$

Now we can construct any of 3^n sequences. If $G(i_1, ..., i_k; j_1, ..., j_m)$ is the sequence with 1's at i_1-th, ..., i_k-th places and 2 at j_1-th, ..., j_m-th places, then

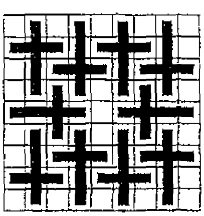
$$G(i_1, ..., i_k; j_1, ..., j_m) = (...((F(i_1, ..., i_k, j_2, ..., i_m; j_1) \oplus F(i_1, ..., i_k, j_1, ..., j_m; j-2) \oplus$$

$$\oplus F(i_1, ..., i_k, j_1, j_2, ..., i_m; j_3) \oplus ... \oplus F(i_1, ..., i_k, j_1, j_2, ..., j_{m-1}; j_m)),$$

as required.

4. Answer : 10.

Note that the angular cell of the board can not belong to a tile. Further, no two neighboring cells on the border of the square can be covered by the tiles. Therefore, at most 16 cells on the border of the square are covered by the tiles. So at most 65 cells can be covered by the tiles (at most 16 are on the border and $7 \cdot 7 = 49$ inside the square). Since every tile consists of 6 cells, we have at most 10 tiles on the board ($\frac{65}{6} < 11$). An example of exactly 10 tiles is shown in the figure.

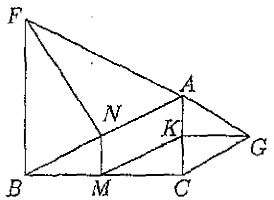


5. Answer: 450.

By condition, $x_1 = \dfrac{p - \sqrt{p^2+4q}}{2}$, $x_2 = \dfrac{p+\sqrt{p^2+4q}}{2}$, so $x_1 < x_2$. Therefore, it suffices to find all pairs (p,q) ($p \in \mathbb{N}$, $q \in \mathbb{N}$) satisfying the inequality $\dfrac{p+\sqrt{p^2+4q}}{2} \le 10$. It is easy to show that there are exactly 450 such pairs.

6. Answer: $BC = 12$.

Let N be the midpoint of AB and K the midpoint of AC (see the Fig.) Then MN and MK are midlines of ΔABC. So $\angle FNM = 90^0 + \angle A = \angle GKM$. Applying the law of cosines to ΔFNM we obtain $11^2 = FN^2 + MN^2 - 2FN \cdot MN \cos \angle FNM$. Similarly, applying the law of cosines to ΔGKM, we obtain $7^2 = GK^2 + MK^2 - 2GK \cdot MK \cos \angle GKM$. Since $FN = \frac{\sqrt{3}}{2}BA$, $GK = \frac{\sqrt{3}}{2}CA$, $MN = \frac{1}{2}CA$, $MK = \frac{1}{2}BA$, we can rewrite these equalities as



$11^2 = \frac{3}{4}BA^2 + \frac{1}{4}CA^2 - 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot BA \cdot CA \cos \angle FNM$, $7^2 = \frac{3}{4}CA^2 + \frac{1}{4}BA^2 - 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot CA \cdot BA \cos \angle GKM$. Subtracting the second equality from the first one gives $72 = 11^2 - 7^2 = (\frac{3}{4} - \frac{1}{4})(BA^2 - CA^2) = \frac{1}{2}(BA^2 - CA^2) = \frac{1}{2}BC^2$, hence $BC^2 = 144$ and finally $BC = 12$.

7. Answer: Jerry wins.

Note that if there are two columns with pawns in the same row, then the player can not put a pawn into the cells of these columns, otherwise the other player can construct the required rectangle. Therefore Jerry has the following strategy to win: he puts the pawn into the cell (the cell must be in the empty column) of the same row in which Tom puts his pawn. So it is easy to see that after Jerry's move Tom can not put the pawn into the cells of occupied columns (i.e. columns containing the pawns). But the number of the columns is even, therefore, Tom loses, since after at most 10 moves all empty columns are exhausted.

8. Multiply the given equality by $abcd$. It is easy to verify that the obtained equality is equivalent to $(b-a)(c-b)(d-c)(a-d) = 0$. Therefore, there are equal numbers among a, b, c, and d.

Category B

1. The required locus is a horizontal line $y = -\frac{1}{4}$ (this line is the directrix of the given parabola).

Let $M(x_0, y_0)$ be the point satisfying conditions. Any (except for vertical) line passing through M can be described by the equation $y = k(x - x_0) + y_0$, where k is a slope. This line touches the parabola if and only if the equation $x^2 = k(x - x_0) + y_0$ has exactly one real root. Therefore, $k^2 - 4x_0k + 4y_0 = 0$. By condition, this equation has two roots k_1 and k_2 (there are two tangents from M to the parabola). Since these tangents are perpendicular we have $k_1k_2 = -1$. By Vieta's theorem $k_1k_2 = 4y_0$, so $y_0 = -\frac{1}{4}$. Hence, $M(x_0, y_0)$ belongs to the required locus if and only if $y_0 = -\frac{1}{4}$, that is M lies on the line $y = -\frac{1}{4}$.

2. Answer: $(m,n) = (0,0), (36,12), (36,18)$.

In any of the cases $m = n, m = n^2, m = -n$ we have $m = n = 0$, and the pair $(0,0)$ is the solution of the given equation.

Let $m + n \neq 0, n \neq 0, m \neq 0$. Adding $4mn(n^2 - m)$ to both sides of the equation we have

$$(m+n)^2(n^2 - m) = 4mn^3. \tag{1}$$

Dividing the initial equation by (1), we have

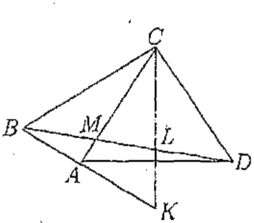
$$\left(\frac{m-n}{m+n}\right)^2 = \frac{m}{n^2}, \tag{2}$$

so m is a perfect square. Let $m = k^2$; suppose for the sake of certaincy that k is a positive integer. From (2) it follows that $\frac{k^2 - n}{k^2 + n} = \pm\frac{k}{n}$. Consider the case $\frac{k^2-n}{k^2+n} = \frac{k}{n}$. This equality can be written as

$$n^2 - (k^2 - k)n + k^3 = 0. \tag{3}$$

Since n is integer, we conclude that the discriminant of the quadratic equation (3) with respect to n must be a perfect square. We have $D = (k^2 - k)^2 - 4k^3 = k^2((k-3)^2 - 8)$. So $(k-3)^2 - 8$ is a perfect square. Let $(k-3)^2 - 8 = l^2, l \geq 0$; then $(k-3-l)(k-3+l) = 8$. Note that the integers $k - 3 - l$ и $k - 3 + l$ have the same parity, therefore from the last equality we obtain either $k - 3 - l = 2$ and $k - 3 + l = 4$ whence $k = 6$ or $k - 3 - l = -4$ and $k - 3 + l = -2$ whence $k = 0$, which contradicts $m \neq 0$. Hence, $k = 6, m = 36; D = 6^2((6-3)^2 - 8) = 36$. For $k = 6$ from (3) it follows that either $n = 12$ or $n = 18$. The case $\frac{k^2-n}{k^2+n} = -\frac{k}{n}$ is considered in the same way. This consideration do not lead to the answers different from $m = 36$ and $n = 12, 18$. It is easy to check that the pairs $(36;12)$ and $(36;18)$ are the solutions of the given equation.

3. Let L be the intersection point of CK and BD. Then $\frac{MC}{ML} = \frac{CD}{LD}$, so that $\frac{MC}{ML} = \frac{MC+CD}{ML+LD} = \frac{MC+CD}{MD}$. By condition, $MA(MC + CD) = MB \cdot MD$, that is $\frac{MB}{MA} = \frac{MC+CD}{MD}$, so $\frac{MB}{MA} = \frac{MC}{ML}$. Then the triangles BMA and CML are similar and therefore $\angle MBA = \angle MCL$. But $\angle MCL = \angle LCD$, thus $\angle KBD = \angle MBA = \angle LCD = \angle KCD$. The angles KBD and KCD are based on the same segment KD and their vertices lie at the same half-plane with respect of this segment. Therefore the points K, B, C, D are concyclic, so $\angle BKC = \angle CDB$, as they are subtended by the same chord BC.



4. Answer: $n = 2$, and -1 is at the midpoint of the side of $\triangle ABC$; or $n = 3$, and -1 is at the center of $\triangle ABC$.

We call a line *labelled* if it passes through the side of any small triangle. Let $F(MN)$ be the operation of changing signs for the point on the labelled line MN.

Suppose that using the described operations we can obtain $+1$ at all labelled points. Note that on any labelled line there is an even number (0 or 2) of the points A, B and C. So the product of the numbers at the points A, B, C are constant under the allowed operations. Since at the final position this product is $+1$, the number -1 can not be at the vertex of $\triangle ABC$.

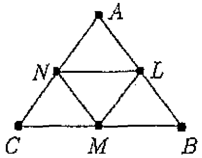


Fig. 1

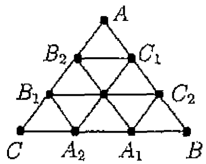


Fig. 2

Let $n = 2$. We have six labelled points (see Fig. 1). Let -1 be at the point M. Then, using $F(BC)$, $F(AC)$, $F(AB)$, $F(LN)$ consecutively, we obtain $+1$ at all labelled points.

Consider the case $n = 3$. We have nine labelled points on the sides of $\triangle ABC$ (see Fig. 2). On any labelled line there are exactly two of these points. Therefore, the product of the numbers at these points is constant. Hence none of these points is labelled by -1. If -1 is at the center of $\triangle ABC$, then using $F(A_2, C_1)$, $F(B_2, A_1)$, $F(C_2, B_1)$, $F(A, B)$, $F(B, C)$, $F(C, A)$ gives $+1$ at all labelled points. Let $n \geq 3$. The number -1 is not at the vertex of $\triangle ABC$. But it is easy to see that any labelled point different from the vertex of $\triangle ABC$, is the vertex of a some regular hexagon with unit side. On any labelled line there is an even number (0 or 2) vertices of these hexagons. So the product of the numbers at the vertices of the hexagons is constant. So there are not -1 at these vertices. Thus for $n \geq 3$ and for any initial position of -1 one can not obtain $+1$ at all labelled points using the allowed operations.

5. Answer: Tom wins.

Note that if there are two columns with pawns in the same row, then the player can not put the pawn into the cells of these columns, otherwise the other player can construct the required rectangle. This statement is valid also for any two rows. We call these columns (rows) *connected*. Further, we call the column (row) *empty*, if it does not contain pawns, and *occupied* in the opposite case. It is easy to see that the player can not put the pawn into the cell on the intersection of occupied row and column.

Tom has the following strategy to win: he puts the pawn into arbitrary cell, so there is an even number of empty columns and empty rows (namely, 24) on the board. Further, after Jerry's move Tom puts his pawn so that after this move the number of empty columns and empty rows on the board will be even again, i.e., if Jerry puts the pawn into the cell on the intersection of empty column and empty row, then Tom also puts the pawn into the cell on the intersection of empty column and empty row; if Jerry puts the pawn into the cell on the intersection of the occupied column (row) and empty row (column), then Tom puts the pawn into the cell on the intersection of this column (row) and empty row (column). Since the number of the moves is finite, it is easy to see that Tom wins.

6. *Lemma.* Let $ABCD$ be a rectangle, X an arbitrary point. Then $XA^2 + XC^2 = XB^2 + XD^2$.

Let O be the center of the rectangle. Applying the law of cosines to the triangles AOX and COX we have

$$XA^2 = AO^2 + XO^2 - 2AO \cdot XO \cdot \cos \angle AOX$$

and

$$XC^2 = CO^2 + XO^2 - 2CO \cdot XO \cdot \cos \angle COX.$$

Summing (taking into account that $\cos \angle COX = -\cos \angle AOX$, $AO = CO = \frac{1}{2}AC$) gives

$$XA^2 + XC^2 = 2AO^2 + 2XO^2 = \frac{1}{2}AC^2 + 2XO^2.$$

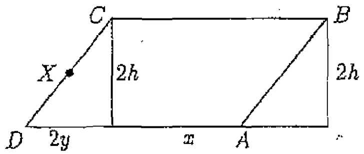
Similarly, $XB^2 + XD^2 = \frac{1}{2}BD^2 + 2XO^2$. It remains to see that $AC = BD$, since the diagonals of the rectangle are equal. Lemma is proved.

Choose the largest of the segments, say, XA. From lemma it follows that all the segments XA, XB, XD are non-degenerated and the inequality $XA^2 \leq XB^2 + XD^2$ holds. Therefore $XA \leq \sqrt{XB^2 + XD^2} < XB + XD$, so that one can construct the triangle with XA, XB, XD as its sides.

b) Let X be the midpoint of the side DC of the parallelogram $ABCD$ (see the fig.). We want to select positive x, y, h so that one can not construct a triangle using three of the segments XC, XD, XB and XA as its sides (see the Fig.). For this to be valid it is sufficient that $2XD = XC + XD < XA$ and $XD + XA < XB$, that is x, y, h satisfy the system

$$2\sqrt{y^2 + h^2} < \sqrt{(x+y)^2 + h^2}, \quad \sqrt{y^2 + h^2} + \sqrt{(x+y)^2 + h^2} < \sqrt{(x+3y)^2 + h^2}.$$

For $h = 0$ (degenerate parallelogram) the first inequality gives $y < x$, and the second one is obviously valid. So we can select any x and y, $0 < y < x$ and $h > 0$ sufficiently close to zero that both the inequalities holds.



Also we can simply set $y = h = \frac{x}{2}$.

7. Answer: $a = b = 1$.

It is evident that $a = b = 1$ is a solution. Suppose now that $a > 1$, $b > 1$. We have $b^b = a^{(a^a)} > a^a$, so $b > a$. Taking into account the equality from the condition, we obtain

$$a^{a-1} < b < a^a. \qquad (*)$$

(Indeed, $a^{(a^a)} = b^b > a^b$, so $a^a > b$; further, $a^{(a^a)} = b^b < (a^a)^b = a^{ab}$, hence $a^a < ab$, or $a^{a-1} < b$.)

Now let p be an arbitrary prime number which divides a and b; set $a = p^s n, b = p^t m$, where n and m are coprime with p. From $(p^s n)^{(a^a)} = (p^t m)^b$ it follows that $sa^a = tb$, whence, in view of $(*)$, $s < t < sa$. So, the exponent t of p in b is less than the exponent sa of p in a^a. Therefore a^a is divisible by b, and $a^a > b$. Let $a^a = kb$, where $k > 1$. Then $b^b = a^{(a^a)} = a^{kb}$, so $b = a^k$. Using $(*)$ we obtain $a^{a-1} < a^k < a^a$, or $a - 1 < k < a$, a contradiction.

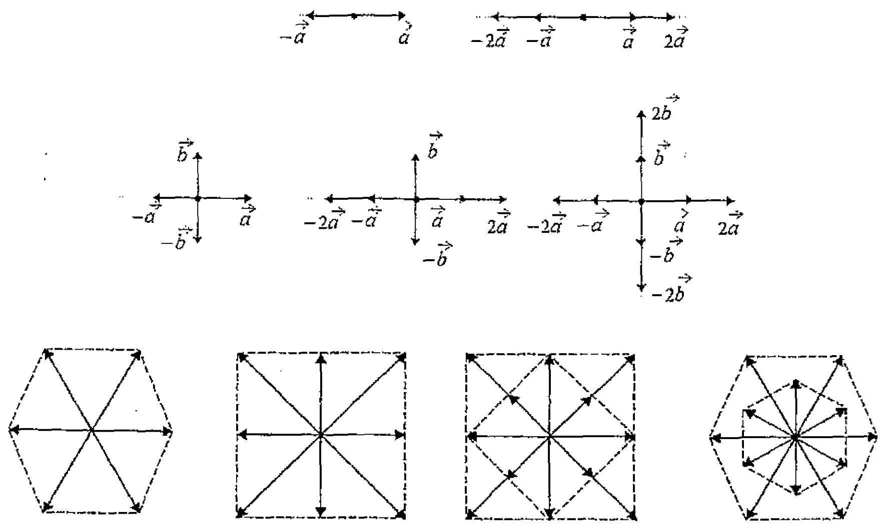
8. Answer: b) there is no infinite concordant set. Twelve vectors.

a) As it immidiately follows from the condition, if $\vec{a} \in R$, then also $-\vec{a} \in R$. Now, for a) to be proved, it is sufficient that for any $\vec{a}, \vec{b} \in \mathbb{R}$ such that $0 < \angle(\vec{a}, \vec{b}) < \frac{\pi}{2}$, the inclusion $\vec{a} - \vec{b} \in \mathbb{R}$ is valid. Let for definiteness $|\vec{a}| \leq |\vec{b}|$. Since $0 < \angle(\vec{a}, \vec{b})$, we have

$$|\vec{a} - S_{\vec{b}}(\vec{a})| < |\vec{a}| + |S_{\vec{b}}(\vec{a})| = |\vec{a}| + |\vec{a}| = 2|\vec{a}| \leq |\vec{a}| + |\vec{b}|.$$

Further, since the vector $\vec{a} - S_{\vec{b}}(\vec{a}) \neq \vec{0}$ and its length is an integer multiple of $\vec{b}$, we have $\vec{a} - S_{\vec{b}}(\vec{a}) = \pm\vec{b}$. The equality $\vec{a} - S_{\vec{b}}(\vec{a}) = -\vec{b}$ is impossible, since $\angle(\vec{a}, \vec{b}) < \frac{\pi}{2}$. Hence $\vec{a} - S_{\vec{b}}(\vec{a}) = \vec{b}$, and then $\vec{a} - \vec{b} = S_{\vec{b}}(\vec{a}) \in R$ as was stated.

b) As for finite concordant sets, all possible cases are shown in the figures below. In particular, the greatest possible number of vectors in finite concordant set is 12.



Category A

1. Answer: a) yes; b) no.

a) Pit can write an arbitrary a and then $c = -1 - a - b$; it is easy to verify that all three equations have the common real root $x = 1$.

b) If Bill writes any b such that $b \neq a$, then the equations have not common root different from 1. Indeed, let k be a common root of the equations and $k \neq 1$.

Subtracting the second equation from the first one gives

$$(a-b)x^2 + (b-c)x + (c-a) = 0, \tag{1}$$

and k satisfies (1). For $a \neq b$ (1) has two roots $x = 1$ and $x = \frac{c-a}{a-b}$, for $a = b \neq c$ (1) has a unique root $x = 1$, and for $a = b = c$ all real numbers satisfy (1). Therefore, since $a \neq b$ and $k \neq 1$, we have

$$k = \frac{c-a}{a-b}. \tag{2}$$

It follows that Pit can not write $c = a$, otherwise $k = 0$, that is the common root of the initial equations is not negative. So, $c \neq a$. Subtracting the first equation from the third one gives

$$k = \frac{b-c}{c-a}. \tag{3}$$

Arguing as above, we conclude that $c \neq b$.

Further, subtracting the third equation from the first one gives

$$k = \frac{a-b}{b-c}. \tag{4}$$

Multiplying (2)-(4), we have $k^3 = 1$ so $k = 1$, a contradiction.

2. Answer: 2400.

It is evident that for $n \geq 15$ the number $\dfrac{n\,!}{2000}$ is integer, that is $\left\{\dfrac{n\,!}{2000}\right\} = 0$, therefore the initial equation has not solution (since by condition q^2 is not integer). Thus $n < 15$. On the other hand, the denominator of the fraction $\dfrac{n\,!}{2000}$ (when reduced to the lowest terms) must be a perfect square. Hence, since $2000 = 2^4 \cdot 5^3$, we see that $n\,!$ is divisible by 5, but it is not divisible by 25 (otherwise the power of 5 in the denominator would be odd). Therefore, $5 \leq n \leq 9$. Now, we have 5 cases.

1. $n = 5$. $\quad n\,! = 2^3 \cdot 3 \cdot 5 \Rightarrow$ the power of 2 in the denominator of the fraction $\dfrac{n\,!}{2000}$ is odd (it equals 1), which is impossible.

2. $n = 6$. $\quad n\,! = 2^4 \cdot 3^2 \cdot 5 \Rightarrow \left\{\dfrac{n\,!}{2000}\right\} = \dfrac{9}{25}$.

3. $n = 7$. $\quad n\,! = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \Rightarrow \left\{\dfrac{n\,!}{2000}\right\} = \dfrac{13}{25}$.

4. $n = 8$. $\quad n\,! = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \Rightarrow \left\{\dfrac{n\,!}{2000}\right\} = \dfrac{4}{25}$.

5. $n=9$. $\quad n! = 2^7 \cdot 3^4 \cdot 5 \cdot 7 \Rightarrow \left\{\frac{n!}{2000}\right\} = \frac{11}{25} = -1 + \frac{36}{25} = \left\{\frac{36}{25}\right\}$.

Therefore, $q = \frac{k}{5}$, where $0 < k < 10000$ since $0 < q < 2000$. Moreover, $\left\{\frac{k^2}{25}\right\} = \left\{\frac{m}{25}\right\}$ is equivalent to $k^2 \equiv m \pmod{25}$. If $m = a^2$, then the congruence $k^2 \equiv a^2 \pmod{25}$ has exactly two solutions $k = a$ and $k = 25 - a$ for $0 < k < 25$, $a \not\equiv 0 \pmod 5$. Besides, this congruence has solutions $k = a + 25l$ and $k = 25 - a + 25l$ for $l = 1, \ldots, 799$. Thus for any $n = 6, 8, 9$ there are exactly 800 solutions of the corresponding equations for $0 < k < 10000$. The congruence $k^2 \equiv 13 \pmod{25}$ has not solution since $k^2 - 13$ is not divisible by 5. Therefore, the initial equation has exactly 2400 solutions.

3. Answer:16.

Let $\mathcal{M}_N$ be the set of all the sequences. The notation $A \sim B$ means that sequence A is similar to B. It is easy to see that $A \sim B$ yields $B \sim A$, and $A \sim C$, $B \sim C$ imply $A \sim B$. Let

$$\Pi(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \Big(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \underbrace{+1, \ldots, +1}_{N-4}\Big),$$

and

$$\mathcal{M}_N^* = \{\Pi(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \mid \varepsilon_i \in \{-1, +1\}, i = 1, 2, 3, 4\}$$

It is easy to show that for any sequence from $\mathcal{M}_N$ there is a similar sequence from $\mathcal{M}_N^*$.

Prove that if the quadruples $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and $\eta_1, \eta_2, \eta_3, \eta_4$ are different then

$$\Pi(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \not\sim \Pi(\eta_1, \eta_2, \eta_3, \eta_4).$$

Let $(e_1, \ldots, e_N) \in \{\mathcal{M}_N\}$. Associate with this sequence the following four numbers

$$I_1 = \prod_{k=0}^{k_1} e_{5k+1}e_{5k+2},\ 5k_1 + 2 \le N, \quad I_2 = \prod_{k=0}^{k_2} e_{5k+1}e_{5k+3},\ 5k_2 + 3 \le N,$$

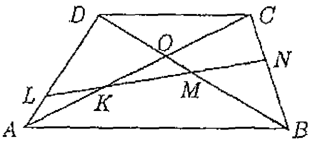
$$I_3 = \prod_{k=0}^{k_3} e_{5k+3}e_{5k+4},\ 5k_3 + 4 \le N, \quad I_4 = \prod_{k=0}^{k_4} e_{5k+4}e_{5k+5},\ 5k_4 + 5 \le N.$$

These numbers are invariant under the allowed operation because the operation changes the sings of exactly two terms for every products, so the products are constant under the operations. For the sequence from $\mathcal{M}_N^*$ we have $I_1 = \varepsilon_1\varepsilon_2$, $I_2 = \varepsilon_2\varepsilon_3$, $I_3 = \varepsilon_3\varepsilon_4$, $I_4 = \varepsilon_4$.

The system $\begin{cases} \varepsilon_1\varepsilon_2 = \omega_1, \\ \varepsilon_2\varepsilon_3 = \omega_2, \\ \varepsilon_3\varepsilon_4 = \omega_3, \\ \varepsilon_4 = \omega_4, \end{cases}$ has the unique solution $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{-1, +1\}$ for any $\omega_1, \omega_2, \omega_3, \omega_4$ from $\{-1, +1\}$. So, the maximal number of sequences from $\mathcal{M}_N^*$ no two of which are similar, is less than or equal to the number of quadruples $(\omega_1, \omega_2, \omega_3, \omega_4)$, that is $2^4 = 16$. But there are exactly 16 sequences in $\mathcal{M}_N^*$, so the proof is complete.

4. Answer: no.

Let O be the intersection point of the diagonals; L, K, M, N (in the given order) intersection points of l with lateral sides and diagonals (see the Fig.). Consider the vectors $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \lambda\vec{a}$, $\overrightarrow{OD} = \lambda\vec{b}$, (here λ is a negative coefficient, $|\lambda| = DC/AB$. Let $\overrightarrow{OK} = t\vec{a}$ and $\overrightarrow{OM} = s\vec{b}$, with t and s (positive) coefficients. We have $\overrightarrow{MK} = t\vec{a} - s\vec{b}$, so,



$$\overrightarrow{DL} = \overrightarrow{DO} + \overrightarrow{OK} + \overrightarrow{KL} = \overrightarrow{DO} + \overrightarrow{OK} + \overrightarrow{MK} = -\lambda\vec{b} + t\vec{a} + (t\vec{a} - s\vec{b}) = 2t\vec{a} - (\lambda + s)\vec{b}.$$

Further, $\overrightarrow{DA} = \overrightarrow{DO} + \overrightarrow{OA} = \vec{a} - \lambda\vec{b}$. Since the vectors $\overrightarrow{DL}$ and $\overrightarrow{DA}$ are collinear, we have $\frac{2t}{1} = \frac{\lambda + s}{\lambda}$ or $\lambda + s = 2\lambda t$. Thus, $LK = KM$ is equivalent to $\lambda + s = 2\lambda t$. Similarly, $KM = MN$ is equivalent to $\lambda + t = 2\lambda s$. Therefore, $LK = KM = MN$ is equivalent to $\begin{cases} \lambda + s = 2\lambda t, \\ \lambda + t = 2\lambda s, \end{cases}$ which is equivalent to the system $\begin{cases} \lambda + s = 2\lambda t, \\ t - s = 2\lambda(s - t). \end{cases}$ From the second equality it follows that either $s = t$ or $2\lambda = -1$. If $s = t$, then the line l is parallel to the bases of the trapezoid. If $2\lambda = -1$, then the last system is equivalent to the system $\begin{cases} \lambda = -\frac{1}{2}, \\ s + t = \frac{1}{2}. \end{cases}$ THis means that the ratio of the lengths of the bases is 1:2, and the points K and M lying on the diagonals satisfy the only condition $\frac{OK}{OA} + \frac{OM}{OB} = \frac{1}{2}$. In this case the line l satisfies the problem condition, but in the case $\frac{OK}{OA} \neq \frac{OM}{OB}$ l is not parallel to the bases.

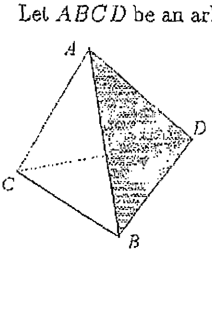
5. Answer: it is possible.

Consider a finite plane over the field F_3 (residue field modulo 3). We assign to each vertex of the given graph some of 9 points on the plane, different points to different vertices. Now, if for some three verices A, B, C the corresponding points lie on the same straight line, then we color the segments AB, BC, AC in the same color. It is easy to see that this is the required coloring.

6. Answer: 1, 2, 3, 4.

We first show that any triangular pyramid has at least one perfect vertex.

Let $ABCD$ be an arbitrary triangle pyramid. Without loss of generality we assume that $AB \geq \max\{AC, AD, BC, BD\}$. For $\triangle ABC$ and $\triangle ABD$ we have $AB < AC + BC$ and $AB < AD + BD$. So, $2AB < AC + BC + AD + BD$, or $2AB < (AC + AD) + (BC + BD)$. From the last inequality it follows either $AB < AC + AD$, or $AB < BC + BD$. If $AB < AC + AD$, then, since $AB \geq AC$ and $AB \geq AD$, one can construct the triangle with AB, AC, AD as its sides; so the vertex A is perfect. Similarly, if $AB < BC + BD$, then the vertex B is perfect. Thus the number N of the



perfect vertices of any triangular pyramid is at least 1.

Now give the examples of the pyramids for which $N = 1, 2, 3, 4$.

For $N = 4$: the right tetrahedron is the required pyramid (all vertices are perfect). For $N = 1$: the pyramid $ABCD$ with $AB = BC = CA = 1$ and $DA = DB = DC$ is the required pyramid (the only perfect vertex is D).

For the case $N = 2$, let $\Delta\, ABC$ be an isosceles right triangle with the hypotenuse AB. Let D be the point not in the plane of $\Delta\, ABC$, such that $|B - D| < \varepsilon$ with sufficiently small ε. Then the pyramid $DABC$ has exactly two perfect vertices: A and C.

Mow for the case $N = 3$, consider a right triangle $\Delta\, ABC$ with $\angle C = 90^0$ such that $\frac{AC}{2} < BC < \frac{AB}{2} < AC$. Let for example $AC = 15$, $BC = 8$, $AB = 17$. Let M be the midpoint of the hypotenuse AB, D a point not in the plane of $\Delta\, ABC$ sufficiently closed to M such that DM is the perpendicular to the plane of $\Delta\, ABC$. Then the lengths of the segments CM, AM, BM are sufficiently closed to $0.5AB = 8.5$. It is easy to see that the pyramid $DABC$ has exactly three perfect vertices A, C, and D.

7. Answer: a) all positive integers $n \neq 2$; b) $a = b = 1$ and $a = 2^8$, $b = 2^{10}$.

a) For $n = 1$ all $a = b \neq 1$ satisfy the condition. If $n > 2$, then the numbers $a = (n-1)^{(n-1)}$ and $b = (n-1)^n$ satisfy the equation (easy verification), and $a > 1$, $b > 1$. It remains to show that $n = 2$ does not satisfy the problem condition. Let $n > 1, a > 1, b > 1$. Let p be a prime divisor of a (and, consequently, p is a prime divisor of b, too). Set $a = p^s l, b = p^t m$, where s, t are positive integers, and positive integers l and m are coprime with p. We have $b^b = (a^a)^n > a^a$, whence $b > a$. So $a^{an} = b^b > a^b$, hence $an > b$. Further, $(p^s k)^{an} = (p^t m)^b$; the exponents of p in both sides of the equality must be equal, so that $san = tb$. Since $b < na$, we have $s < t$. Therefore, for any p the exponent s of p in a is less than the exponent t of p in b. Thus a divides b, or $b = ka$ $(k > 1)$. It follows that $b^b = (ka)^{ka} = a^{na}$, whence $(ka)^k = a^n$, i.e. $a^{n-k} = k^k > 1$ or $n - k > 0$. Since $k > 1$, we have $n > 2$.

b) Similar arguments apply to the case $n = 5$: $b = ka$ $(k > 1$, and $(ka)^k = a^5$, i.e. $a^{5-k} = k^k$, $1 < k < 5$. It is easy to verify that there are not solutions if either $k = 2$ or $k = 3$. For $k = 4$ we obtain $a = 4^4 = 2^8, b = 4a = 2^{10}$.

8. Answer: 4.

1) We construct a non-isosceles triangle ABC such that there are at most 4 distinct numbers in the set of lengths and angles of this triangle. Let $AB = \pi/2\,(m)$, $\angle A = \pi/2\,(rad)$, and $\angle C = \alpha$, where $\alpha \tan\alpha = \pi/2$. This α exists, because, if $f(x) = x\tan x$, then $f(0) = 0$ and $f(x) \to +\infty$, as $x \to \pi/2$, (it is evident that $\alpha \neq \pi/4$). Therefore $AC = BC \cos\alpha = \cos\alpha AB/\sin\alpha = \alpha$. Thus the required triangle is constructed.

2) Suppose, contrary to our claim, that there exists a non-isosceles triangle $\Delta\, ABC$, such that in its set $(a, b, c, \alpha, \beta, \gamma)$ there at most 3 distinct numbers. Since $\Delta\, ABC$ is non-isosceles, we have $a \neq b \neq c \neq a$ and $\alpha \neq \beta \neq \gamma \neq \alpha$. Without no loss of generality we assume that $a < b < c$. So $\alpha < \beta < \gamma$. Since there are at most 3 distinct numbers among the numbers a, b, c, α, β, γ, we have $a = \alpha$, $b = \beta$, $c = \gamma$. By the law of sines, $a/\sin a = b/\sin b = c/\sin c$. But the function $g(x) = x/\sin x$ is increasing on the interval $(0, \pi)$, so it follows that $a = b = c$, a contradiction.

SELECTION and TRAINING SESSION

Test 1

1. Answer: 9 (see the example in Fig. 1)

Consider the following cross-like tiles as on Fig. 2. Any of the tiles must contain at least a painted cell (or, if not, the central cell of the tile would not satisfy the condition). Hence, the number of painted cells must be greater than or equal to 8. Also, one can show that this number can not be equal to 8.

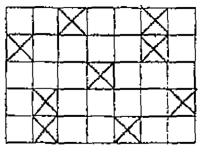


Fig. 1

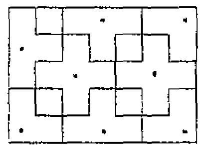
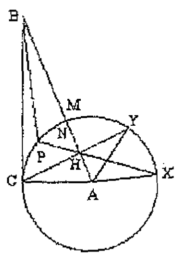


Fig. 2

2. (Proof of Siarhei Markouski.) Let S be a circle centered at A, of radius $AC = AP$; let N be the intersection point of S and AB. Produce CH and PH to meet S again at Y and X respectively (see the Fig.). Since $CH \perp AB$, we have $CH = HY$. By the power-of-a-point theorem for S, $PH \cdot HX = CH \cdot HY = CH^2$. Now, from the right triangle ABC we have $CH^2 = BH \cdot HA$, so that $PH \cdot HX = BH \cdot HA$. It follows that the points B, P, A, X are concyclic. Hence, $\angle XAH = \angle XPB$. Furhter, $\angle XPN = \frac{1}{2}\angle XAN$ since $\angle XPN$ subtends the arc NYX. So, $\angle XPN = \frac{1}{2}\angle XPB$, whence $\angle HPN = \angle BPN$, i.e. PN is the bisector of $\angle BPN$. That is, PM is the bisector of $\angle BPH$ if and only if



$$M = N \iff AM = AN \iff AM = AC \iff \angle A = 60^0.$$

3. Suppose that there exists such a function. For $n \geq 2$ we have

$$f(n+1) - f(n) = f(f(n-1)) > 0, \quad f(n+1) > f(n).$$

Hence for any $n \geq 2$ the function $f(n)$ increases. It follows that

$$f(n) \geq f(n-1) + 1 \geq ... \geq f(2) + (n-2) \geq n-1, \quad f(n) \geq n-1$$

for $n \geq 2$. Now , for $n \geq 4$ we have $f(n-1) \geq 2$, hence we may apply the last inequality to get $f(f(n-1)) \geq f(n-1) - 1$. Thus (for $n \geq 4$)

$$f(n+1) = f(f(n-1)) + f(n) \geq (f(n-1)-1) + n - 1 = f(n-1) + (n-2) \geq (n-1) - 1 + (n-2),$$

$$f(n+1) \geq 2n - 4.$$

In particular, (setting $n = 7$) $f(8) \geq 10$. Let $r = f(8)$. Now, summing the equalities $f(f(n-1)) = f(n+1) - f(n)$ for $n = 9, ..., r-1$ $(r-1 \geq 9)$ we get

$$f(f(8)) + [f(f(9)) + ... + f(f(r-2))] = f(r) - f(9), \; [f(f(9)) + ... + f(f(r-2))] = -f(9),$$

a contradiction. (In the case$r - 1 = 9$ the expression in square brackets means 0.)

Hence, there is not a function satisfying the given equation.

4. Denote the given pentagonal line by $A_1A_2A_3A_4A_5$, and the center of the sphere by O. On the contrary, suppose that $l = A_1A_2 = ... = A_5A_1 > \sin 72^0$. Then $\angle A_1OA_2 = \angle A_2OA_3 = ... = \angle A_5OA_1 > 144^0$. From the trihedral cone $OA_1A_2A_3$ we have $\angle A_1OA_3 + \angle A_1OA_2 + \angle A_2OA_3 \leq 360^0$ whence $\angle A_1OA_3 < 360^0 - 2 \cdot 144 = 72^0$. Similarly, $\angle A_3OA_5 < 72^0$. Further, from the trihedral cone $OA_1A_3A_5$ we get $\angle A_1OA_5 \leq \angle A_1OA_3 + \angle A_3OA_5$. Thus $144^0 < \angle A_1OA_5 \leq \angle A_1OA_3 + \angle A_3OA_5 < 72^0 + 72^0 = 144^0$, a contradiction.

Test 2

1. Answer: 8.

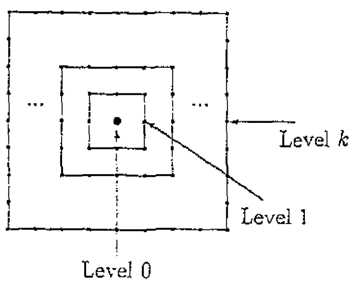
Let n be the number of vertices of the given polyhedron. Since the degree of any vertex is 4, the total number of the edges of the polyhedron equals $m = \frac{4n}{2} = 2n$. If k is the total number of the faces, then, by Euler's theorem, $k = 2 + m - n = 2 + 2n - n = n + 2$. Let a be the number of triangular faces, then the number of other faces is $k - a = n + 2 - a$, any of them containing at least 4 edges. So the number of the edges

$$2n = m \geq \frac{3a + 4(n+2-a)}{2}, \quad 4n \geq 3a + 4(n+2-a),$$

whence $a \geq 8$. Now for an octahedron the value $a = 8$ holds.

2. (Solution of Artsiom Kokhan, Jury Hrushetski, Dzmitry Doryn.) Consider the function in x: $f(x) = 2x^3 - 3(a+b)x^2 + 6abx + 2a^3 - b^3 - 6a^2b + 3ab^2$. We have $f'(x) = 6x^2 - 6(a+b)x + 6ab = 6(x-a)(x-b)$. Since $a < b$ and $f(x)$ is cubic polynomial, the leading coefficient being $2 > 0$, we conclude that $x = a$ is a point of local maximum of $f(x)$ and $x = b$ is a point of local minimum of $f(x)$. Substituting $x = a$, we easily find that $f(a) = (a-b)^3 < 0$. Hence, for any $x \leq b$ we have $f(x) \leq f(a) < 0$, that is the only zero of $f(x)$, namely $x = c$, satisfies $c > b$.

3. We partition the set of all integer points into an infinite collection of subsets which we call the subset of level 0, of level 1, of level 2, and so on. It is easy to see now that we can construct the required sequence as follows. First, we write the point $(0, 0)$ (i.e. the point of level 0). Then we pass to the subset of level 1 and write in an appropriate succession all 8 points of this level. Then we pass to the level 2, and so on.



4. Answer: $\dfrac{|b^2-a^2|}{4(a^2+b^2)}$.

Test 3

1. Answer: $\lambda \in (-\infty, \frac{1}{2}] \cup [\frac{3}{2}, +\infty)$.

(Solution of Jury Hrushetski, Maxim Zhyhovich, Aliaxei Kirkouski, Siarhei Markouski.) We use the following formulae for the median $m_a^2 = \frac{1}{4}(2b^2+2c^2-a^2)$. Note that in any triangle ABC with M the midpoint of BC, $AC - CM < AM < AC + CM$, that is $b - \frac{a}{2} < m_a < b + \frac{a}{2}$. Similarly, $a - \frac{b}{2} < m_b < a + \frac{b}{2}$. Summing the above inequalities, we get

$$\frac{a+b}{2} < m_a + m_b < \frac{3}{2}(a+b). \tag{1}$$

Suppose now that $a \neq b$ and the equality

$$m_a + \lambda a = m_b + \lambda b \tag{2}$$

holds. Then $\lambda = \dfrac{m_a - m_b}{b-a} = \dfrac{m_a^2 - m_b^2}{(b-a)(m_a+m_b)} = \dfrac{3}{4}\dfrac{b^2-a^2}{(b-a)(m_a+m_b)} = \dfrac{3}{4}\dfrac{b+a}{m_a+m_b}$. Now from (1) it follows that $\frac{2}{3} < \dfrac{a+b}{m_a+m_b} < 2$, so that $\frac{3}{4}\cdot\frac{2}{3} < \lambda < \frac{3}{4}\cdot 2$, or $\frac{1}{2} < \lambda < \frac{3}{2}$. So, if $\lambda \in (-\infty, \frac{1}{2}] \cup [\frac{3}{2}, +\infty)$, and (2) holds, then a must equal b.

It remains to show that for any $\lambda \in (\frac{1}{2}, \frac{3}{2})$ there exists a triangle with $a \neq b$ satisfying (2) or, equivalently, $\lambda = \dfrac{m_a - m_b}{b-a}$.

Consider all the triangles ABC with fixed sides $CA = b$, $CB = a$ ($\frac{b}{2} < a < 2b$), and variable angle $x = \angle ACB$, $0 < x < \pi$; set $f(x) = \dfrac{m_a - m_b}{b-a}$, where m_a, m_b are the medians of the corresponding triangle. It is easy to see that $m_a \to b - \frac{a}{2}$ as $x \to 0$, and $m_a \to b + \frac{a}{2}$ as $x \to \pi$. So,

$$f(x) \to \frac{(b-\frac{a}{2}) - (a - \frac{b}{2})}{b-a} = \frac{3}{2} \quad \text{as } x \to 0, \text{ and } f(x) \to \frac{(b+\frac{a}{2}) - (a+\frac{b}{2})}{b-a} = \frac{1}{2} \quad \text{as } x \to \pi.$$

Hence, for any $\lambda \in (\frac{1}{2}, \frac{3}{2})$ there exists an $x \in (0,\pi)$ such that $f(x) = \lambda$. This is what was to be shown.

2. a) Let $[x]$ be the integer part of x. We have for any $n \in \mathbb{N}$ $[n\sqrt{3}] < n\sqrt{3}$, i.e. $[n\sqrt{3}]^2 < 3n^2$, $[n\sqrt{3}]^2 \le 3n^2 - 1$. Now, the equation $x^2 = 3y^2 - 1$ has no integer solutions, hence we have in fact $[n\sqrt{3}]^2 \le 3n^2 - 2 = 3n^2\left(1 - \frac{2}{3n^2}\right) < 3n^2\left(1 - \frac{1}{3n^2}\right)^2$, so that $[n\sqrt{3}] < n\sqrt{3}\left(1 - \frac{1}{3n^2}\right) = n\sqrt{3} - \frac{1}{n\sqrt{3}}$, and finally $\frac{1}{n\sqrt{3}} < \{n\sqrt{3}\}$.

b) Answer: no.

Let $c > 1$ be any constant. We show that there exists $n \in \mathbb{N}$, for which

$$\{n\sqrt{3}\} < \frac{c}{n\sqrt{3}}. \tag{1}$$

First we consider the equation

$$x^2 - 3y^2 = -2. \tag{2}$$

We claim that (2) has infinitely many solutions $x, y \in \mathbb{N}$. Indeed, let $(1+\sqrt{3})(2+\sqrt{3})^k = a_k + \sqrt{3}\, b_k$ for positive integer k; $a_k, b_k \in \mathbb{N}$. Then also $(1-\sqrt{3})(2-\sqrt{3})^k = a_k - \sqrt{3}\, b_k$, and we have $a_k^2 - 3b_k^2 = (1+\sqrt{3})(1-\sqrt{3}) \cdot ((2+\sqrt{3})(2-\sqrt{3}))^k = -2 \cdot 1^k = -2$. It remains to note that a_k and b_k indefinitely increase as k increases.

Further, if $(x, y) \in \mathbb{N} \times \mathbb{N}$ is a solution of (2), then obviously $x < y\sqrt{3}$. Also $y\sqrt{3} - 1 < x$, because $(y\sqrt{3} - 1)^2 < x^2 = 3y^2 - 2 \quad \Leftrightarrow \quad \frac{\sqrt{3}}{2} < y$ which is true. Hence, $x = [y\sqrt{3}]$.

Now, for (1) to be valid, we choose a solution $(x, y) \in \mathbb{N} \times \mathbb{N}$ of (2) with sufficiently large y such that $y^2 > \dfrac{c^2}{6(c-1)}$, and $y^2 > \dfrac{c}{3}$ or $y\sqrt{3} > \dfrac{c}{y\sqrt{3}}$. Then put in (1) $n = y$, so that $[n\sqrt{3}] = x$. Then (1) becomes

$$\{y\sqrt{3}\} < \frac{c}{y\sqrt{3}} \quad \Leftrightarrow \quad y\sqrt{3} - x < \frac{c}{y\sqrt{3}} \quad \Leftrightarrow \quad \left(y\sqrt{3} - \frac{c}{y\sqrt{3}}\right)^2 < x^2 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{c^2}{3y^2} - 2c < x^2 - 3y^2 \quad \Leftrightarrow \quad y^2 > \frac{c^2}{6(c-1)},$$

so (1) is valid.

3. (Solution of Aliaksandr Kirkouski, Artsiom Kokhan.) First construct a graph with 90 edges satisfying the problem condition. Let $A_1, \ldots, A_{15}$ be the vertices. Let the vertices A_i and A_j be connected with blue edge if and only if $i - j \equiv 1$ or $4 \pmod 5$, with red one if and only if $i - j \equiv 2$ or $3 \pmod 5$, and be not connected if and only if $i - j \equiv 0 \pmod 5$. It is easy to verify that this graph is required.

Now prove that in any bichromatic graph with 15 vertices and at least 91 edges there exists a monochromatic triangle. We use the following

Lemma. Let in a graph with k vertices among any three vertices there be some two which are connected with an edge. Then the total number of edges of the graph is at least $\dfrac{k(k-2)}{4}$.

Indeed, let $p(k)$ be the minimal possible number of edges in the graph. Consider an arbitrary pair of non-connected vertices. Any of the remaining $k-2$ vertices is connected with at least one of these two. Also, the graph with these $k-2$ vertices has at least $p(k-2) \geq \dfrac{(k-2)(k-4)}{4}$ (induction assumption) edges. Hence, $p(k) \geq p(k-2) + (k-2) \geq \dfrac{k(k-2)}{4}$ as was stated.

Also we may conclude that if in a graph with k vertices among any three vertices at least two are not connected, then there are at least $\frac{k(k-2)}{4}$ pairs of vertices in the graph which are not connected.

Now let Γ be a graph with at least 91 edges. Then there exists a vertex A of degree more than or equal to 13. Let A belong to r red and b blue edges ($r+b \geq 13$). Let AR_1, ..., AR_r be red edges, AB_1, ..., AB_b be blue ones. Consider the graph with vertices R_1, ..., R_r. It does not contain red edges (otherwise, we would have a red triangle of the type AR_iR_j). Hence, all the edges are blue, and among any three vertices at least two are not connected. Thus there are at least $\frac{r(r-2)}{4}$ pairs of vertices which are not connected. Similarly, at least $\frac{b(b-2)}{4}$ pairs of B_1, ..., B_b are not connected. Thus the total number of pairs of non-connected vertices in the graph is at least $\frac{b(b-2)}{4}+\frac{r(r-2)}{4}=$ $=\frac{b^2+r^2-2(b+r)}{4} \geq \frac{(b+r)^2-4(b+r)}{8}=\frac{(b+r)(b+r-4)}{8} \geq \frac{13\cdot 9}{8} > 14$, hence at least 15. So, the total number of the edges of the graph is at most $\frac{15\cdot 14}{2}-15=90$, a contradiction.

Test 4

1. Answer: $f(x)=C_1$, $g(x)=C_2$, $h(x)=C_1+C_2$ for any constants $C_1, C_2 \in \mathbb{R}$. (Solution of Siarhei Markouski.) 1) Put $x=0$ in

$$f(x+y^3)+g(x^3+y)=h(xy), \tag{1}$$

then $g(y)=h(0)-f(y^3)$. Thus (1) becomes

$$f(x+y^3)+h(0)-f((x^3+y)^3)=h(xy) \tag{2}$$

2) Put $y=-x^3$ in (2). Then

$$f(x-x^9)+h(0)-f(0)=h(-x^4). \tag{3}$$

Further, replacing x by $-x$ we have

$$f(x^9-x)+h(0)-f(0)=h(-x^4) \iff f(x-x^9)=f(x^9-x).$$

Since x^9-x takes all real values, we conclude that f is even, $f(-x)=f(x)$.

Prove now that for any $C<0$ there exist x_0 and y_0 such that $x_0y_0=C$ and $(x_0^3+y_0)^3=-(x_0+y_0^3)$, that is $(x_0^3+\frac{C}{x_0})^3=-(x_0+\frac{C^3}{x_0^3})$, or, equivalently, $P(x_0)=$ $=x_0^{12}+3x_0^8C+3x_0^4C^2+x_0^4+2C^3=0$. Note that $P(0)=2C^3<0$ and $P(x)\to+\infty$ as $x\to+\infty$, so that the required x_0 really exists.

4) Put $x=x_0$, $y=y_0$ in (2), then using that $f(x_0+y_0^3)=f(-(x_0+y_0^3))$ we get $h(C)\equiv h(0)$ for all real $C\leq 0$. In particular, $h(-x^4)\equiv h(0)$, and (3) yields

$f(x - x^9) \equiv f(0)$. Since $x^9 - x$ takes all real values, $f(x) \equiv f(0)$. Let $f(0) = C_1$. Then $g(x) = h(0) - f(x^3) = h(0) - C_1 = C_2$, and $h(x) \equiv C_1 + C_2$.

2. (Solution from the IMO-99 Shortlist.) *Lemma.* If M is an interior point of the convex quadrilateral $ABCD$ then $MA + MB < AD + DC + CB$.

Proof. The ray AM intersects the quadrilateral in N; suppose, for instance, that $N \in [CD]$ (see Fig. 1). Then $MA + MB < MA + MN + NB \leq AN + NC + CB \leq AD + DN + NC + CB = AD + DC + CB$. Lemma is proved.

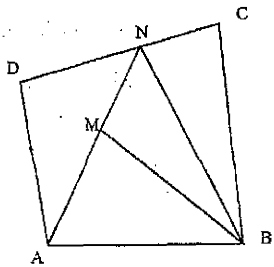


Fig.1

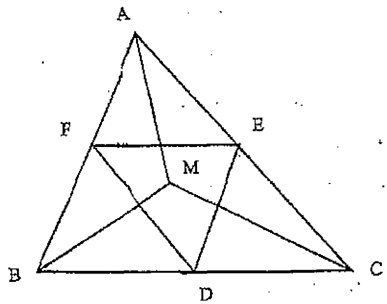


Fig. 2

The median triangle DEF divides triangle ABC into four regions (see Fig. 2). Each region is covered by at least two of the convex quadrilaterals $ABDE$, $BCEF$, $CAFD$. If, for instance, M belongs to $[ABDE]$ and $[BCEF]$ then $MA + MB < BD + DE + EA$ and $MB + MC < CE + EF + FB$. By adding these two inequalities we get $MB + (MA + MB + MC) < AB + BC + CA$, which implies the required conclusion.

3. Suppose a contrary. Let N be a positive integer such that for all $a_0 \in \mathbb{N}$ and $d \in \mathbb{N}$, there exist n for which $S(a_n) \leq N$. Set $d = 10^m + 1$. Consider

$$a_n = a_0 + n(10^m + 1) = \sum_{k=0}^{s} b_k 10^k, \quad b_k \in \{0, 1, ..., 9\}; \tag{1}$$

by assumtion $\sum_{k=0}^{s} b_k \leq N$.

It is easy to see that $10^l \equiv 10^k \ (mod\ (10^m + 1))$ where $k \equiv l \ (mod\ 2m)$, hence from $a_n \equiv a_0 \ (mod\ (10^m + 1))$ and (1) it follows that

$$a_0 \equiv a_n = \sum_{k=0}^{s} b_k 10^k \equiv \sum_{k=0}^{2m-1} c_k 10^k \ (mod\ (10^m + 1)), \quad \sum_{k=0}^{2m-1} c_k \leq N. \tag{2}$$

The total amount of different numbers of the type $\sum_{k=0}^{2m-1} c_k 10^k$ equals exactly

$$K_{N,2m} = \binom{2m + N}{N} = \frac{(2m + N)!}{N!(2m)!} = \frac{(2m + N)(2m + N - 1) \cdot \ldots \cdot (2m + 1)}{N!}.$$

(Note that the equality $K_{l,n} = C^n_{l+n}$ easily follows by induction, for instance, in n with using the obvious recurrence relation $K_{l,n+1} = K_{l,n} + K_{l-1,n} + ... + K_{1,n} + 1$.) It is easy to see that there exists a positive integer m such that $K_{N,2m} < 10^m + 1$, therefore there exists a positive integer a_0 such that (2) does not hold. This contradiction does prove the statement.

Test 5

1. a) Let R be the circumradius of $\triangle ABC$. We have

$$a\tan\frac{\alpha}{2} = \frac{a\sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}} = \frac{2a\sin^2\dfrac{\alpha}{2}}{2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}} = \frac{a(1-\cos\alpha)}{\sin\alpha} = 2R(1-\cos\alpha),$$

and

$$a\cot\frac{\alpha}{2} = \frac{a\cos\dfrac{\alpha}{2}}{\sin\dfrac{\alpha}{2}} = \frac{2a\cos^2\dfrac{\alpha}{2}}{2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}} = \frac{a(1+\cos\alpha)}{\sin\alpha} = 2R(1+\cos\alpha).$$

Hence we need to prove the inequalities

$$\begin{aligned} R(1-\cos\alpha) \le m_a \le R(1+\cos\alpha), &\quad \text{if} \quad \alpha < \frac{\pi}{2}; \\ R(1+\cos\alpha) \le m_a \le R(1-\cos\alpha), &\quad \text{if} \quad \alpha > \frac{\pi}{2}. \end{aligned} \tag{1}$$

Let O be the circumcenter of $\triangle ABC$. Then, obviously,

$$AO - OM \le AM \le AO + OM, \tag{2}$$

where

$$AM = m_a, \quad AO = R, \quad OM = \begin{cases} R\cos\alpha & \text{if} \quad \alpha < \dfrac{\pi}{2}, \\ R\cos(\pi-\alpha) = -R\cos\alpha & \text{if} \quad \alpha > \dfrac{\pi}{2}. \end{cases}$$

Now, substituting the expressions for AM, AO, OM in (2), we get the required (1).

b) Let K be the intersection point of AL and the sircumcircle ($K \ne A$). Then $\angle CAK = \angle KAB$ implies $KC = KB$, $KM \perp BC$, and $\angle CKM = \dfrac{\pi}{2} - \dfrac{\alpha}{2}$. Since $KM \le KL$ and $AK \le 2R$, we have

$$AL = AK - KL \le 2R - KM. \tag{3}$$

Substituting $KM = \dfrac{a}{2}\tan\dfrac{\alpha}{2} = R(1-\cos\alpha)$ in (3), we get $l_a = AL \le 2R - R(1-\cos\alpha) = R(1+\cos\alpha)$. It remains to note that $R(1+\cos\alpha) = \dfrac{a}{2}\cot\dfrac{\alpha}{2}$.

2. (Solution from the IMO-99 Shortlist.) Let $n = 2^a5^bp$, where a, b are non-negative integers and $(p, 10) = 1$. We notice that it is enough to find M such that $p|M$ and the sum

of M's digits is k (we can take then $m = M \cdot 10^c$, where $c = max\{a, b\}$). Since $(p, 10) = 1$, there exists $k \geq 2$ such that $10^k \equiv 1 \pmod p$. It follows that $10^{ik} \equiv 1 \pmod p$ and $10^{jk+1} \equiv 10 \pmod p$ for every non-negative integers i, j. We will look for integers $u, v \geq 0$ so that $M = \sum_{i=1}^{u} 10^{ki} + \sum_{j=1}^{v} 10^{kj+1}$ (if u or v is 0 then the corresponding sum is 0).

Notice that $M \equiv u + 10u \pmod p$. Hence M is acceptable if

$$\begin{cases} u + v = k, \\ p | u + 10v, \end{cases} \iff \begin{cases} u + v = k, & (1) \\ p | k + 9v. & (2) \end{cases}$$

Because $(p, 3) = 1$, one of the residues (mod p) of the numbers k, $k + 9$, $k + 18, \ldots$, $k + 9(p-1)$ must be nil, so relation (2) holds for some v_0 with $0 \leq v_0 < p$. Taking this v_0 and $u_0 = k - v_0$ we get the wanted M.

3. (Solution from the IMO-99 Shortlist.) Suppose that there exists a coloring function $f : \mathbb{Z} \to \{R, B, G, Y\}$ with the property that for any integer a

$$f\{a, a + x, a + y, a + x + y\} = \{R, B, G, Y\}.$$

In particular, coloring the integer lattice by the rule

$$g : \mathbb{Z} \times \mathbb{Z} \to \{R, B, G, Y\}, \quad g(i, j) = f(ix + jy),$$

we obtain that the vertices of any unit square have all four different colors.

Claim 1. If there exists a column $i \times \mathbb{Z}$ such that $g|i \times \mathbb{Z}$ is not periodical with period 2, then there exists a row $\mathbb{Z} \times j$ such that $g|\mathbb{Z} \times j$ is periodical with period 2.

Proof. If $g|i \times \mathbb{Z}$ is not periodical, then we can find the configuration $\begin{matrix} Y \\ B \\ R \end{matrix}$; using the adjacent unit squares, we get $\begin{matrix} RYR \\ GBG \\ YRY \end{matrix}$ and also $\begin{matrix} YRYRY \\ BGBGB \\ RYRYR \end{matrix}$ and so on. Thus we obtain three periodical lines.

Claim 2. If for one integer i, $g_i = g|\mathbb{Z} \times i$ is periodical with period 2, then for every $j \in \mathbb{Z}$, $g_j = g|\mathbb{Z} \times j$ has period 2. The values of g_i are the values of g_j if $i \equiv= J \pmod 2$ and the other two values if $i \not\equiv j \pmod 2$.

Proof. Applying the square rule to the line $\ldots RBRBRB \ldots$ we get

$$\begin{matrix} \ldots YGYGY \ldots \\ \ldots RBRBR \ldots \end{matrix} \quad \text{and next} \quad \begin{matrix} \ldots RBRBR \ldots \\ \ldots YGYGY \ldots \\ \ldots RBRBR \ldots \end{matrix} \quad \text{or} \quad \begin{matrix} \ldots BRBRB \ldots \\ \ldots YGYGY \ldots \\ \ldots RBRBR \ldots \end{matrix}$$

A similar argument holds for the rows below the line $\mathbb{Z} \times i$.

Changing between them 'rows' and 'columns' we have similar claims. So we can suppose that the rows are periodical with period 2 and $g(0, 0) = R$, $g(1, 0) = B$. Therefore $g(y, 0) = B$ (y is odd). The row $\mathbb{Z} \times \{x\}$ is odd too; hence $g(\mathbb{Z} \times \{x\}) = \{Y, G\}$. From $g(y, 0) = f(xy) = g(0, x)$ we get a contradiction.

1. Answer: $n = 4$.

(Solution of Siarhei Markouski.) It is easy to verify that the subsets $M_1 = \{3k+1\} \cap M$, $M_2 = \{9k+2, 9k+3\} \cap M$, $M_3 = \{27k+5, 27k+6\} \cap M$, $M_4 = \{14, 15, ..., 26, 27\}$ satisfy the condition.

Now suppose that M is partitioned into $k < 4$ subsets satisfying the condition. We may assume that $k = 3$, so that $M = M_1 \sqcup M_2 \sqcup M_3$. Since $|M| = 40$, at least one of M_i's contains more than or equal to 14 elements, e.g. $|M_1| \geq 14$. Let $a_1 < a_2, ... < a_{14}$ be some 14 elements from M_1. Consider the numbers

$$b_1 = a_{14} - a_1,\ b_2 = a_{14} - a_2,\ ...,\ b_{13} = a_{14} - a_{13}\ \in M.$$

None of b_i belongs to M_1 (otherwise we would have $a_{14} = a_i + b_i$, $a_{14}, a_i, b_i \in M$, contrary to the condition). Hence $b_i \in M_2 \sqcup M_3$. So one of M_2, M_3 contains at least 7 of b_i. Let for definiteness $b_1, b_2, ..., b_7 \in M_2$, b_1 being the largest of b_i's. Consider the numbers

$$c_7 = b_1 - b_7,\ c_6 = b_1 - b_6,\ \in M.$$

In similar way, neither c_7 nor c_6 belong to M_2. Moreover, none of them belongs to M_1 (otherwise we would have $c_i = b_1 - b_i = (a_{14} - a_1) - (a_{14} - a_i) = a_i - a_1$, $c_i + a_1 = a_i$ and c_i, a_1 a_i belong to M_1, contrary to the condition). So both c_7, c_6 belong to M_3. Let $c = c_6 - c_7$. Similarly we conclude that c can not belong to any of M_1, M_2 M_3, a contradiction.

2. (Solution of Alexandr Usnich.) Consider the following numbers: $x_n = \dfrac{5 \cdot 10^{n-1} + 1}{3}$, $y_n = \dfrac{10^n + 2}{3}$. We claim that x_n^2 and y_n^2 are monotonic, x_n^2 having $2n - 1$ digits, and y_n^2 having $2n$ digits. Indeed,

$$x_n^2 = \frac{25 \cdot 10^{2n-2} + 10^n + 1}{9} = \frac{2(10^{2n-1} - 1)}{9} + \frac{5(10^{2n-2} - 1)}{9} + \frac{10^n - 1}{9} + 1 =$$

$$= \underbrace{22...2}_{2n-1} + \underbrace{55...5}_{2n-2} + \underbrace{11...1}_{n} + 1 = \underbrace{27...78...89}_{2n-1}$$

if $n \geq 3$ and $x_1^2 = 4$, $x_2^2 = 289$;

$$y_n^2 = \frac{10^{2n} + 4 \cdot 10^n + 4}{9} = \frac{10^{2n} - 1}{9} + \frac{4(10^n - 1)}{9} + 1 = \underbrace{11...1}_{2n} + \underbrace{44...4}_{n} + 1 = \underbrace{1...15...56}_{2n}$$

if $n \geq 2$, and $y_1^2 = 16$.

3. (Solution of Alexandr Usnich.)

Answer: no.

If we set $(1, 0) = \vec{a}$ and $(2, 1) = \vec{b}$, then the problem is whether we can obtain the pair $(\vec{a}, \vec{b})$ from the pair $(\vec{a}, \vec{b} - 2\vec{a})$ if the first operation is of the type A. That is, in the coordinates of the basis $\vec{a}, \vec{b}$, whether we can obtain the pair $((1, 0), (0, 1))$ from the pair $((1, 0), (-2, 1))$. We now prove that it is not possible.

Denote by $((a_n, b_n), (c_n, d_n))$ the pair obtained on the n-th step. Let also $((a_0, b_0), (c_0, d_0))$ be the initial pair. Since the first operation is of the type A, the type of operation performed on any odd step is A, while the type of operation performed on any even step is B.

So,

$$a_{2n+1} = a_{2n} + 2k_{2n+1}c_{2n}, \quad b_{2n+1} = b_{2n} + 2k_{2n+1}d_{2n}, c_{2n+1} = c_{2n}, \quad d_{2n+1} = d_{2n},$$

and

$$a_{2n} = a_{2n-1}, \quad b_{2n} = b_{2n-1}, \quad c_{2n} = c_{2n-1} + 2k_{2n}a_{2n-1}, \quad d_{2n} = d_{2n-1} + 2k_{2n}b_{2n-1},$$

where $k_{2n+1} \neq 0 \neq k_{2n}$ are some integers. We claim that $|a_{2n}| < |c_{2n}|$, $|b_{2n}| < |d_{2n}|$, and $|a_{2n+1}| > |c_{2n+1}|$, $|b_{2n+1}| > |d_{2n+1}|$. This is true for $n = 0$. Let the inequalities hold for $n = s$. Then for $n = s + 1$ we have $|c_{2s+2}| = |c_{2s+1} + 2k_{2s+2}a_{2s+1}| \geq 2|k_{2s+2}| \cdot |a_{2s+1}| - |c_{2s+1}| \geq 2|a_{2s+1}| - |c_{2s+1}| > 2|a_{2s+1}| - |a_{2s+1}| = |a_{2s+1}| = |a_{2s+2}|$ (here we used that $k_{2s+2} \neq 0$. All the other inequalities for $n = s + 1$ can be proved similarly.

Now, if we obtain the pair $((1, 0), (0, 1))$ on $2n$-th step, then $1 = |a_{2n}| < |c_{2n}| = 0$ — a contradiction. If we obtain the pair on $(2n + 1)$-th step, then $0 = |b_{2n+1}| > |d_{2n+1}| = 1$ — a contradiction again.

b) Answer: all pairs $((a, b), (c, d))$, for which

$$ad - bc = 1 \tag{1}$$

and

$$a \equiv d \equiv 1 \pmod 4, \quad b \equiv c \equiv 0 \pmod 2. \tag{2}$$

The conditions (1) and (2) are necessary (trivial induction).

Now prove that (1) and (2) are sufficient. First note that $((a, b), (c, d))$ can be obtained from $((1, 0), (0, 1))$ if and only if $((1, 0), (0, 1))$ can be obtained from $((a, b), (c, d))$.

So, let $((a, b), (c, d))$ be a pair satisfying (1), (2). Consider the following cases.

1) $|ab| = |cd|$. Then from (1) $(a, b) = (a, c) = (d, c) = (d, b) = 1$ ((a, b) means G.C.D.(a, b)), hence $|a| = |d|$, $|b| = |c|$. Then from (1), (2) it easily follows $((a, b), (c, d)) = ((1, 0), (0, 1))$. That is, there is nothing to do in this case.

2) $|ab| < |cd|$. Then inequalities $|a| \geq |c|$, $|b| \geq |d|$ can not hold simultaneously.

a) Let $|a| < |c|$. Choose $n \in \mathbb{Z}$ such that $|c + 2na| \leq |a|$, n being obviously nonzero. Now perform the operation A with $k = n$ thus obtaining $a' = a$, $b' = b$, $c' = c + 2na$, $d' = d + 2nb$. Then a', b', c', d' satisfy (1), (2), and the inequality $|c'| \leq |a'|$. Hence $|a'd'| = |b'c' + 1| \leq |b'| \cdot |c'| + 1 \leq |b'| \cdot |a'| + 1$, and $|a'|(|d'| - |b'|) \leq 1$. As it follows from (2), $a' \neq 0$. Thus $|d'| \leq |b'| + 1$. If $|d'| = |b'| + 1$ then $|a'| = 1$, hence $|c'| \leq 1$, or, in view of (2), $|a'| = 1$, $|c'| = 0$ and then from (1) $|d'| = 1$. Finally, $|a'| = |d'| - 1 = 0$, $|b'| = 0$ and we are done. If $|d'| \leq |b'|$, we get $|c'd'| \leq |a'b'|$.

b) Let $|b| < |d|$. Then choose $n \in \mathbb{Z}$ such that $|d + 2nb| \leq |b|$, $n \neq 0$, and perform the operation A with $k = n$ obtaining a', b', c', d' as above with $|d'| \leq |b'|$. Now we have $|b'c'| = |a'd' - 1| \leq |a'| \cdot |d'| + 1 < |a'| \cdot |b'| + 1$. The last equality is strong since $|d'| \neq |b'|$, $|a'| \neq 0$ in view (2). Thus $|b'|(|c'| - |a'|) < 1$, and $|b'| > |d'| \geq 1$, so that $|c'| \leq |a'|$, and we have again $|c'd'| \leq |a'b'|$.

In case $|c'd'| = |a'b'|$ there is nothing to do. In case 3)$|c'd'| \leq |a'b'|$ we proceed similarly as in case 2), the operation being used is of type B. Continuing the process, we get on some step $|c'd'| = |a'b'|$, and at this moment we have that the pair $((a',b'),(c',d'))$ is $((1,0),(0,1))$. Thus the assertion is proved.

Test 7

1. (Solution of Aliaksandr Kirkouski.) Set $m = \dfrac{a}{\sqrt[3]{x}}$, $n = \dfrac{b}{\sqrt[3]{y}}$, $k = \dfrac{c}{\sqrt[3]{z}}$. Then the required inequality becomes

$$3(m^3+n^3+k^3)(x^3+y^3+z^3) \geq (mx+ny+kz)^3. \tag{1}$$

Note that

$$3(m^3+n^3+k^3) = (1^2+1^2+1^2)(m^3+n^3+k^3) \geq (m^{3/2}+n^{3/2}+k^{3/2})^2 \tag{2}$$

by the Cauchy-Schwarz-Buniakovski inequality.

Applying the Hölder inequality

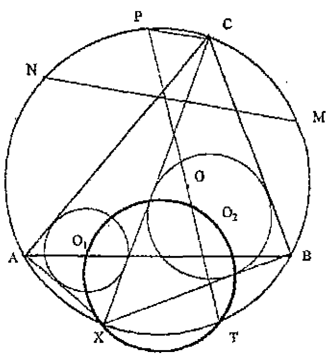
$$(m^p+n^p+k^p)^{1/p}(x^q+y^q+z^q)^{1/q} \geq mx+ny+kz$$

with $p=2/3$, $q=1/3$ $\ (1/p+1/q=1)$ and cubing both sides, we get

$$(m^{3/2}+n^{3/2}+k^{3/2})^2(x^3+y^3+z^3) \geq (mx+ny+kz)^3. \tag{3}$$

Multiplying (2) and (3) we have the required inequality (1).

2. (Solution from the IMO-99 Shortlist.) Let M be the midpoint of the arc BC (see the Fig.). It is a known result (otherwise easy to prove) that triangle MO_2B is isosceles with $MO_2 = MB$. Therefore $MO_2 = MB = MC = MO$, where O is the incenter of the triangle ABC. In the same way, A, O_1, O, C are on a circle with center N, the midpoint of the arc AC. Let P be on Ω such that $CP \parallel MN$ and T be the second intersection of OP with Ω. Then $MO = MC = NP$ and $MP = NC = NO$, therefore $MPNO$ is a parallelogram. It follows that $S(MPT) = S(NPT)$, whence $MP \cdot MT = NP \cdot NT$. Thus $NC \cdot MT = MC \cdot NT$, which shows that $NT : NO_1 = MT : MO_2$. But then $\Delta NO_1T \sim \Delta MO_2T$ (because $\angle O_1NT = \angle XNT = \angle XMT = \angle O_2MT$), $\angle NTO_1 = \angle MTO_2$, $\angle O_1XO_2 = \angle MTN = \angle O_1TO_2$ therefore the quadrilateral XO_1O_2T is cyclic.



This proves that the circle (XO_1O_2) passes through the fixed T for any position of X.

3. (Solution from the IMO-99 Shortlist.) A game with n players is determinated by ordering the $N = \binom{n}{2}$ transpositions (i,j) of the set $\{1,\ldots,n\}$: $t_1,\ldots,t_N$. The game is nice if the permutation $P = t_N \circ \ldots \circ t_2 \circ t_1$ has no fixed point and it is tiresome if P is the identity.

Claim 1. There exists a nice game with n players if and only if $n \neq 3$.

If $n = 3$ and the players are denoted such that $t_1 = (a, b)$, $t_2 = (a, c)$, $t_3 = (b, c)$, then $P = t_3 t_2 t_1 = (a, c)$ has a fixed point.

For every n and the changes made in the order (1,2), (1,3),..., (1,n), (2, 3), (2,4),..., (2, n),..., $(n-1, n)$ we get, using induction

$$P = (n-1,n)(n-2,n)(n-2,n-1)...(2,3)(1,n)(1,n-1)...(1,3)(1,2) =$$

$$= \begin{pmatrix} 2 & 3 & ... & i & ... & n \\ n & n-1 & ... & n-i+2 & ... & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & ... & n \\ 2 & 3 & ... & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 2 & ... & i & ... & n \\ n & n-1 & ... & n-i+1 & ... & 1 \end{pmatrix}$$

and, for $n = 2k$, we have no fixed point: $i \neq 2k - i + 1$.

For $n = 2k + 1$, $k \geq 2$, prolong the previous ordering by

$$(1, 2k+1), (2, 2k+1), ..., (k, 2k+1), (2k, 2k+1), (2k-1, 2k+1), ..., (k+1, 2k+1):$$

$$P = (k+1, 2k+1)...(2k, 2k+1)(k, 2k+1)...(1, 2k+1)(2k-1, 2k)...(1,2) =$$

$$= \begin{pmatrix} 1 & 2 & ... & k-1 & k & k+1 & k+2 & ... & 2k & 2k+1 \\ 2 & 3 & ... & k & 2k & 2k+1 & k+1 & ... & 2k- & 1 \end{pmatrix} \circ$$

$$\circ \begin{pmatrix} 1 & 2 & ... & k-1 & k & k+1 & ... & 2k \\ 2k & 2k-1 & ... & k+2 & k+1 & k & ... & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 2 & ... & k-1 & k & k+1 & k+2 & ... & 2k & 2k+1 \\ 2k-1 & 2k-2 & ... & k+1 & 2k+1 & 2k & k & ... & 2 & 1 \end{pmatrix}$$

has no fixed points ($k + 1 \neq 2k$ because $k \neq 1$).

Claim 2. There exists a tiresome game with n players if and only $n = 4k$ or $n = 4k+1$.

Indeed, the signature of the permutation P is equal to $(-1)^{\binom{n}{2}}$ and thus n must be $4k$ or $4k + 1$. If $n = 4$ we have the solution:

$$(3,4)(1,3)(2,4)(2,3)(1,4)(1,2) = \text{id}. \tag{1}$$

Between two groups of four players we choose the partial game:

$$(4,7)(3,7)(4,6)(1,6)(2,8)(3,8)(2,7)(2,6)(4,5)\cdot$$

$$\cdot(4,8)(1,7)(1,8)(3,5)(3,6)(2,5)(1,5) = \text{id}. \tag{2}$$

With (1) and (2) we can build a tiresome $4k$-game: divide the $4k$ players into k groups, each of them containing 4 players; then play the game (1) inside each group and the game (2) between any two different groups, in an arbitrary order.

An identity $4k+1$-game can be obtained from a $4k$-game by inserting in each block of moves of type (1) the transpositions with $4k + 1$ such that the block is not disturbed:

$$[(3,5)(3,4)(4,5)](1,3)(2,4)(2,3)(1,4)[(1,5)(1,2)(2,5)] =$$
$$= (3,4)(1,3)(2,4)(2,3)(1,4)(1,2) = \text{id}.$$

Test 8

1. (Solution of Alexandr Usnich.) Let $AB = AC = BD = p$, $PC = a$, $PD = b$; R be the circumradius of ΔABP. Then $BP = p - b$, $AP = p - a$. By the power-of-a-point theorem, $DP \cdot DB = DO^2 - R^2$, or $bp = DO^2 - R^2$. Similarly, $ap = CO^2 - R^2$. It follows that $DO^2 - CO^2 = p(b - a)$. Since ΔABD is isosceles $(AB = BD)$, we have $DI = AI$. In similar way, $CI = BI$. Further, if T is the point of tangency of AB and the incircle of ΔABP, then

$$AT = \frac{p + (p-a) - (p-b)}{2} = \frac{p+b-a}{2} \text{ and } BT = \frac{p+(p-b)-(p-a)}{2} = \frac{p+a-b}{2}.$$

Now

$$DI^2 - CI^2 = AI^2 - BI^2 = (AT^2 + TI^2) - (BT^2 + TI^2) = AT^2 - BT^2 =$$

$$= (AT - BT)(AT + BT) = p(b-a),$$

whence $DO^2 - CO^2 = DI^2 - CI^2$.

Note that the locus of points X satisfying the equation $DX^2 - CX^2 = const \neq 0$ is the couple of straight lines, each perpendicular to CD. One of the lines intersects the segment CD at some inner point X_0, the other intersects the line CD outside the segment CD at some point X_1. We have to show that O and I lie on the same line. For this to be done we begin to move the point C continuously keeping $AC = p$ till the moment when $ABCD$ becomes symmetric (i.e. $ABCD$ becomes an equilateral trapezoid). At this moment O and I lie on the same line passing through the midpoints of AB and CD. Hence in the initial quadrilateral the points O and I lie on the same line, so OI and DC are perpendicular.

2. (Solution from the IMO-99 Shortlist.) We use the following

Lemma. If $a, c \in \mathbb{N}$ and $a^2 | c^2 + 1$ then there exists $b \in \mathbb{N}$ such that $a^2(a^2+1) | b^2 + 1$.

Proof. Indeed $a^2 | (c + a^2c - a^3)^2 + 1$ and $a^2 + 1 | (c + a^2c - a^3)^2 + 1$, so we can take $b = c + a^2c - a^3$.

Using the lemma we see that it is enough to find strictly increasing sequences (a_n) and (c_n) such that $a_n^2 | c_n^2 + 1$ for every $n \in \mathbb{N}$. This can be realised by taking for instance $a_n = 2^{2n} + 1$, $c_n = 2^{na_n}$. In this case

$$c_n^2 + 1 = (2^{2n})^{a_n} + 1 = (a_n - 1)^{a_n} + 1$$

is divisible by a_n^2.

3. Set $f(x, y) = x^2 - xy + y^2$. Suppose that such a partition exists; let A, B, C be the subsets of the partition.

Lemma 1. If $x = y + z$ for some positive integers x, y, z, then x, y, z can not belong to three different subsets.

Indeed, let $x \in A, y \in B, z \in C$. It is easy to verify the identity $f(a + b, a) = f(a + b, b)$. Hence, $f(x, y) \in C$, and, on the other hand, $f(x, y) = f(y + z, y) = f(y + z, z) = f(x, z) \in B$, a contradiction.

Now, let for definiteness $1 \in A$, $b < c$ be the minimal elements in B and C, respectively. Then we have $1, \ldots, b - 1 \in A$.

Lemma 2. The subset C contains some multiple kb of b ($k \geq 2$).

There is nothing to prove if $c \vdots b$. Let $c \not\vdots b$. Consider $b-1$ elements $c-r$, where $r = 1, \ldots, b-1$. Since c is minimal in C, we have $c - r \notin C$. Further, $c-r \notin B$, otherwise we get $r + (c-r) = c$, where $r \in A, c-r \in B, c \in C$, contrary to lemma 1. Thus $c - r \in A$. Since $c \not\vdots b$, among the elements $c-1, \ldots, c-(b-1)$ there is one which is a multiple of b: $c - r = nb$. Hence, $f(nb, b) \in C$ is a multiple of b in C, and lemma 2 is proved.

Let now kb be the least multiple of b containing in C. Then $(k-1)b \notin C$. Further, $(k-1)b \notin A$, otherwise $(k-1)b + b = kb$, contrary to lemma 1. Hence, $(k-1)b \in B$.

Lemma 3. For any positive integer n we have $(nk-1)b+1 \in A$ and $nkb+1 \in A$.

We use induction on n. Let $n = 1$. Then $(k-1)b+1 \notin C$, since $1 \in A, (k-1)b \in B$. Also $(k-1)b+1 \notin B$, since $b-1+((k-1)b+1) = kb$, and $b-1 \in A, kb \in C$. Hence, $(k-1)b+1 \in A$. Now, $kb+1 \notin C$, since $kb+1 = (k-1)b+1+b$, and $(k-1)b+1 \in A, b \in B$. Finally, $kb+1 \notin B$, since $kb \in C$ and $1 \in A$. Hence, $kb+1 \in A$.

Suppose that $((n-1)k-1)b+1 \in A$ and $(n-1)kb+1 \in A$. We have $(nk-1)b+1 \notin C$, since $(nk-1)b+1 = (n-1)kb+1+(k-1)b$, and $(n-1)kb+1 \in A, (k-1)b \in B$. Further, $(nk-1)b+1 \notin B$, since $(nk-1)b+1 = ((n-1)k-1)b+1+kb$, and $((n-1)k-1)b+1 \in A, kb \in C$. So, $(nk-1)b+1 \in A$. Now, $nkb+1 \notin C$, since $nkb+1 = (nk-1)b+1+b$, and $(nk-1)b+1 \in A, b \in B$. Also, $nkb+1 \notin B$, since $nkb+1 = (n-1)kb+1+kb$, and $(n-1)kb+1 \in A, kb \in C$. Hence, $nkb+1 \in A$. Lemma 3 is proved.

Now, to obtain the desired contradiction, consider the numbers $kb+1 \in A$ and $kb \in C$. We have $f(kb+1, kb) \in B$. But $f(kb+1, kb) = f(kb+1, 1) = (kb+1)^2 - (kb+1) + 1 = kb(kb+1) + 1 = mkb + 1$ for $m = kb+1$. Then by lemma 3 $mkb+1 \in A$.

Thus the statement is proved.

List of Symbols

$[x]$ — the greatest integer not exceeding x
$\{x\} = x - [x]$
$f: L \to K$ — a function f is defined on the set L and takes its values in the set K
$|M|$ — the number of elements of a finite set M
$b \vdots a$ or $a \mid b$ — b is divisible by a
S_P, $S(P)$ — the area of the figure P

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L БЕЛАРУСКАЯ МАТЭМАТЫЧНАЯ АЛІМПІЯДА

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